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Norm-Constrained Choices

Abstract: This paper develops a general and unified framework to discuss individual choice behaviors that are constrained by the individual's internalized norms. We propose a new notion of rationalizablity of a choice function that incorporates such constraints, and axiomatically study several norm-constrained choice behaviors.

0. Introduction

When making choices from various situations, an individual is often confronted with different *internalized norms* that put constraints on his choices. For example, when presented with the last remaining apple in the fruit basket at a dinner table, the individual may refrain from picking this last apple due to an internalized rule of good behavior–never picking the last apple in the basket (Sen (1993)). In a similar fashion, when faced with various sized slices of a birthday cake on a table, the individual refrains from picking *the largest* slice of cake since the individual has a "principle learned at his mother's knee: 'never pick the largest slice of cake'" (Sen 1993). Norm-constrained choice behaviors have been observed in other settings as well. For example, Rabin (1993) reports that "a consumer may not buy a product sold by a monopolist at an 'unfair' price, even if the material value to the consumer is greater than the price", and Sen (1988) observes that an individual chooses to read the official, government newspaper when there are several daily newspapers available and yet *refuses* to read the same official newspaper when other newspapers except the official one are banned by the government (see also Gaertner and Xu 2004). In these settings, by refusing to choose the only available option contained in the feasible set, the individual expresses a protest arising from procedural considerations[1]: when the procedure "producing" the feasible set is viewed as unacceptable, the individual should register a protest by refusing to choose anything from this feasible set. These norm-constrained choice behaviors are not just confined to some isolated examples. Indeed, in the Chinese culture, according to Confucius' teaching, an

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[1] See, for example, Hansson 1996, Suzumura 1999, and Suzumura and Xu 2001; 2003; 2006 for other procedural considerations in choice situations.

individual when making choices should always strike a balance and choose the median.

The norm-constrained choices cannot be easily explained by the conventional choice theory, since they violate some very basic properties (e.g., the standard contraction property requiring a chosen alternative from a set continue to be chosen when the set shrinks and the initially chosen one is still available after the shrinkage, and the non-emptiness of a choice set) for choices to be rationalizable by a well-behaved preference relation. In the literature, there have been formal attempts to study such norm-constrained choices (see, for example, Baigent and Gaertner 1996, and Gaertner and Xu 1997; 1999a; 1999b; 2004). In each of those studies, a specific framework is developed for a specific norm and there seems a lack of a general and unified approach to study such norm-constrained behaviors. The main purpose of this paper is to propose and develop a *single and unified* framework that enables us to study norm-constrained choice behaviors. The thrust of our framework is the following. The individual is modeled as making sequential choices: first, the individual uses his *internalized norms* to put constraints on feasible sets by *rejecting* alternatives that are deemed to be *non-choosable* or *unacceptable* according to the norms; then, the individual chooses the "best" alternative among those surviving the first stage's rejection according to a well-behaved binary relation. It is clear that our framework is general enough to include the conventional choice theory as a special case in which the individual does not reject any alternative in the first stage.

The structure of the remaining parts of the paper is as follows. In Section 1, we lay down our basic framework by presenting notation and definitions. Section 2 is devoted to axiomatic properties observed choices may have. Results are contained in Section 3. We conclude the paper in Section 4 by making several observations.

1. Notation and Definitions

Let X be the finite set of alternatives. The elements in X will be denoted by x, y, etc. Let $\mathcal{K}$ be the set of all subsets, including the empty set, of X. The elements of $\mathcal{K}$ will be denoted by A, B, etc. A choice function C is a mapping from $\mathcal{K}$ to $\mathcal{K}$ such that, for all $A \in \mathcal{K}$, $C(A) \subseteq A$. Note that $C(\emptyset) = \emptyset$ and that we allow $C(A) = \emptyset$ for some non-empty $A \in \mathcal{K}$.[2] For simplicity, in this paper, we confine our attention to the case in which $\#C(A) \leq 1$ for all $A \in \mathcal{K}$; that is, $C(A)$ is either a singleton or the empty set.

Let R be a linear order (reflexive, transitive, complete and anti-symmetric) over X. For all $A \in \mathcal{K}$, if $A \neq \emptyset$, then we define $G(A, R) =: \{x \in A | xRy \ \forall y \in A\}$, and if $A = \emptyset$, then we define $G(\emptyset, R) = \emptyset$. For any given linear order R, P is to

[2]The interpretation of an empty choice set is the following. When $A \neq \emptyset$ and $C(A) = \emptyset$, we may say that the individual chooses *nothing* from A. In the standard literature of the theory of choice, it is often assumed that $C(A) \neq \emptyset$ for any non-empty A. Aizerman and Aleskerov 1995 is an exception.

denote its asymmetric part (xPy iff xRy and not(yRx)) and I is to denote its symmetric part (xIy iff $x = y$).

A *norm*, to be denoted by N, is a mapping from $\mathcal{K}$ to $\mathcal{K}$ such that, for all $A \in \mathcal{K}$, $N(A) \subseteq A$. The interpretation of a norm N is the following. For any non-empty $A \in \mathcal{K}$, $N(A)$ is regarded as the *unacceptable* alternatives of A according to the individual's internalized norm; or, alternatives in $N(A)$ are not choosable by the individual when the feasible set of alternatives is given by A.

In this paper, we consider the following four different internalized norms. First, for a given linear order R over X and for any $A \in \mathcal{K}$ with $\#A \geq 1$, we shall write A as $\{a_1, a_2, \ldots, a_{\#A}\}$ with $a_1 R a_2 R \ldots R a_{\#A}$, and $a_1, \ldots, a_{\#A}$ being distinct. Now, a norm $N : \mathcal{K} \to \mathcal{K}$ is said to be

a modesty-based norm, to be denoted by $N_{modesty}$, if there exists a linear order over X such that, for all $A \in \mathcal{K}$ with $\#A \geq 1$, $N(A) = \{a_1\}$;

a median-based norm, to be denoted by N_{median}, if there exists a linear order over X such that, for all $A \in \mathcal{K}$ with $\#A \geq 2$, $N(A) = \{a_1, \ldots, a_{\#A/2}\}$ if $\#A$ is even and $N(A) = \{a_1, \ldots, a_{(\#A-1)/2}\}$ if $\#A$ is odd;

a protest-based norm, to be denoted by $N_{protest}$, if for all $A \in \mathcal{K}$, $N(A) = \{a \in A : C(\{a\}) = \emptyset\}$;

a weak protest-based norm to be denoted by $N_{w.protest}$, if for all $A \in \mathcal{K}$, $N(A) = A$ if $C(\{a\}) = \emptyset$ for all $a \in A$, and $N(A) = \emptyset$ if $C(\{a\}) \neq \emptyset$ for some $a \in A$.

We now introduce a general notion of rationalizability of a choice function. A choice function C is said to be *norm-constrained rationalizable* if there exist a linear order R over X and a norm N over $\mathcal{K}$ such that, for all $A \in \mathcal{K}$, $C(A) = G(A \setminus N(A), R)$.

Combined with our four norms introduced earlier, we have the following notions of norm-constrained rationalizability of a choice function: A choice function C is said to be

(i) *modesty-norm-constrained rationalizable*, if there exist a linear order R over X and a norm N such that, for all $A \in \mathcal{K}$, $N(A) = N_{modesty}(A)$ and $C(A) = G(A \setminus N(A), R)$;

(ii) *median-norm-constrained rationalizable* if there exist a linear order R over X and a norm N such that, for all $A \in \mathcal{K}$, $N(A) = N_{median}(A)$ and $C(A) = G(A \setminus N(A), R)$;

(iii) *protest-norm-constrained rationalizable* if, if there exist a linear order R over X and a norm N such that, for all $A \in \mathcal{K}$, $N(A) = N_{protest}(A)$ and $C(A) = G(A \setminus N(A), R)$;

(iv) *weakly protest-norm-constrained rationalizable* if, if there exists a linear order R over X and a norm N such that, for all $A \in \mathcal{K}$, $N(A) = N_{w.protest}(A)$ and $C(A) = G(A \setminus N(A), R)$.

It may be noted that the notion of modesty-norm-constrained rationalizability is similar to the notion of rationalizability used by Baigent and Gaertner (1996). The only difference is that, in theirs, $N(\{x\}) = \emptyset$ for any $x \in X$ while in ours, $N(\{x\}) = \{x\}$ for any $x \in X$. Our notion of median-norm-constrained rationalizability is similar to the notion of rationalizability of a choice function by the median studied in Gaertner and Xu (1999b). The difference is that, in their framework, the choice set of a feasible set containing an even number of alternatives contains the two "median" alternatives in the set, while in our approach, the choice for such a feasible set is always the "lower-median" alternative. For example, suppose x, y, z and w are ordered ascendingly. Then, according to Gaertner and Xu (1999b), $C(\{y, z\}) = \{y, z\} = C(\{x, y, z, w\})$, while according to our notion of the median-norm-constrained rationalizability, $C(\{x, y, z, w\}) = \{y\} = C(\{y, z\})$. Our protest-norm-constrained rationalizability and weakly protest-norm-constrained rationalizability correspond, respectively, to the P-rationalizability and LP-rationalizability studied in Gaertner and Xu (2004), though, in their framework, Gaertner and Xu (2004) take norms as given, and in our framework, norms are *revealed*. Finally, we may note that the standard rationalizability of a choice function becomes a special case of our norm-constrained rationalizability in which $N(A) = \emptyset$ for all $A \in \mathcal{K}$.

2. Axioms

Emptiness of Singleton Choice Situations (ESCS): For all $x \in X$, $C(\{x\}) = \emptyset$.

Non-Emptiness of Singleton Choice Situations (NESCS): For all $x \in X$, $C(\{x\}) = \{x\}$.

Non-Emptiness of Non-singleton Choice Situations (NENCS): For all $A \in \mathcal{K}$ with $\#A \geq 2$, $C(A) \neq \emptyset$.

Non-Emptiness of Non-Protest Situations (NENPS): For all $A \in \mathcal{K}$, if there exists $x \in A$ such that $C(\{x\}) \neq \emptyset$, then $C(A) \neq \emptyset$.

Constrained Contraction Consistency (CCC): For all $A \in \mathcal{K}$ with $\#A \geq 3$, there exists $a^* \in A$ with $\{a^*\} \neq C(A)$ such that, for all $A_1, A_2 \subseteq A$, if $a^* \in A_1 \subseteq A_2$ and $C(A_2) \subseteq A_1$, then $C(A_1) = C(A_2)$.

Restricted Contraction (RC): For all $A, B \in \mathcal{K}$ and all $x \in X$, if $x \in B \subseteq A$, then $[C(A) \subseteq B$ and $C(B) \neq \emptyset] \Rightarrow C(A) = C(B)$.

Anti-Unanimity (AU): For all distinct $x, y, z \in X$, if $\{x\} = C(\{x, y\}) = C(\{x, z\})$, then $\{x\} \neq C(\{x, y, z\})$.

Consistency of a Revealed Norm (CRN): For all $A \in \mathcal{K}$, and all $x \in A$, if $\{x\} \neq C(A)$ and $\{x\} \neq C(\{x \cup C(A)\})$, then, for all $y \in A \setminus \{x\}$, $\{x\} \neq C(\{x, y\})$.

Independence of Rejected Alternatives (IRA): For all $A \in \mathcal{K}$ with $\#A \geq 3$, there exist distinct $x, y \in A$ such that $C(A) \cap \{x, y\} = \emptyset$ and $C(B) = C(B \setminus \{x, y\})$ for all $B \subseteq A$.

Minimal Consistency of Rejection (MCR): For all distinct $x, y, z \in X$, if $\{x\} \neq C(\{x, y, z\})$ and $\{x\} \neq C(\{x, y\})$, then $\{x\} \neq C(\{x, z\})$.

Strong Protest Consistency (SPC): For all $x \in X$, all $A \in \mathcal{K}$, if $x \notin C(\{x\})$, then $x \notin C(A \cup \{x\})$.

Weak Protest Consistency (WPC): For all $A \in \mathcal{K}$, all $x \in X \setminus A$, if $C(\{x\}) = \emptyset$ and $C(A) = \emptyset$, then $x \notin C(A \cup \{x\})$.

ESCS stipulates that the choice set of a singleton feasible set is *always* empty. As the discussion in the Introduction may indicate, when an individual has a rule of good behavior, or a sense of protest, this may happen. NESCS, on the other hand, requires that the choice set of a singleton feasible set is *always* non-empty. This is in accordance with the conventional choice behavior. NENCS simply requires that every non-empty and non-singleton feasible set has a non-empty choice set. This again is the standard assumption made for conventional choice behavior. NENPS requires the choice set of a feasible set containing at least one 'non-protest' alternative (the choice set from the singleton set containing this alternative is the singleton set itself) be non-empty.

CCC is a restricted version of the conventional contraction property and requires the conventional contraction property to hold with reference to a fixed alternative–as long as this fixed alternative continues to be available after a shrinkage of a set, the choice from the larger set should coincide with the choice from the smaller set. The fixed alternative can thus be viewed as a reference alternative based on which the choice is made from a set of available alternatives. Take, for example, the choice behavior of never choosing the uniquely largest slice of cake. The reference alternative in this kind of behavior is the largest slice of cake–as long as the largest slice continues to be available, the choice behavior would be similar to the conventional one. RC is yet another restricted version of the conventional contraction property and requires a chosen alternative from a set A continue to be chosen when A shrinks to a subset B as long as the initial chosen alternative is still available in B and the choice set of B is non-empty.

AU stipulates that, if x is the uniquely chosen alternative in pairwise comparison between x and y, and between x and z, then x should not be chosen from the larger set containing x, y and z. A similar but slightly stronger axiom than AU is proposed in Baigent and Gaertner (1996). AU is in sharp contrast with the conventional axiom of γ (see, for example, Sen 1977), which requires that an alternative a be chosen from the set containing a, b and c whenever a is chosen in pairwise comparisons between a and b and between a and c. The idea that AU tries to capture seems to be that norms are context dependent. When, for example, x is the smaller slice of cake in comparisons with each of any other slices of cake on the table, a norm based on modesty would call for the individual to pick up the smaller one over the larger one if they are the only two slices on offer. However, when presented with three slices, by not picking up x, the smallest slice, the individual can still behave modestly.

CRN requires that, if x is not chosen from a set A and from a set containing the chosen alternative from A and x, then x should not be chosen from any doubleton set containing any alternative from A and x. In some sense, CRN reflects a consistency of a revealed norm. IRA says that, for any set A containing three or more alternatives, there always exist two alternatives such that each is not the chosen alternative from A and the deletion of them from A will not affect choices. MCR stipulates that, if x is not the chosen alternative from the set $\{x, y, z\}$ and from the set $\{x, y\}$, then x cannot be chosen from the set $\{x, z\}$. SPC requires that a protest alternative will never be chosen from any set A. WPC is weaker than SPC and requires that the addition of a protest alternative to a set A will not make it choosable as long as the choice set of A is empty.

3. Results

Theorem 1. A choice function C is *modesty-norm-constrained rationalizable* if and only if it satisfies axioms ESCS, NENCS, CCC, AU and CRN.

Proof. It can be checked that if a choice function C is modesty-norm-constrained rationalizable, then it satisfies axioms ESCS, NENCS, CCC, AU and CRN. Therefore, we have only to show that if a choice function C satisfies ESCS, NENCS, CCC, AU and CRN, then it is modesty-norm-constrained rationalizable.

Let C be a choice function that satisfies ESCS, NENCS, CCC, AU and CRN. By NENCS, for all distinct $x, y \in X$, $C(\{x, y\}) \neq \emptyset$. Define the binary relation R over X as follows: for all distinct $x, y \in X$, xIx, and xPy iff $\{y\} = C(\{x, y\})$. Clearly, R is reflexive, complete and anti-symmetric. To show that R is transitive, we consider distinct $x, y, z \in X$ such that xPy and yPz. From the definition of R, we must have $\{y\} = C(\{x, y\})$ and $\{z\} = C(\{y, z\})$. We need to show that xPz, that is, $\{z\} = C(\{x, z\})$. Suppose to the contrary that $\{z\} \neq C(\{x, z\})$. By NENCS, it must be true that $\{x\} = C(\{x, z\})$. By NENCS, $C(\{x, y, z\}) \neq \emptyset$. Let $C(\{x, y, z\}) = \{a\}$, where $a \in \{x, y, z\}$. By CCC, there exists $a^* \in \{x, y, z\}$ such that $a^* \neq a$ and $\{a\} = C(\{a^*, a\})$. Consider the the following three cases that exhaust all possibilities: (i) $a^* = x$; (ii) $a^* = y$; and (iii) $a^* = z$.

(i) $a^* = x$. In this case, by CCC, $a = y$ or $a = z$. If $a = y$, that is, $C(\{x, y, z\}) = \{y\}$, then, by CRN, noting that $x \neq y = C(\{x, y, z\})$ and $\{x\} \neq C(\{x, y\})$, we would have $C(\{x, z\}) \neq \{x\}$, a contradiction. If $a = z$, that is, $C(\{x, y, z\}) = \{z\}$, then, by CCC, it would follow that $C(\{x, z\}) = \{z\}$, a contradition.

(ii) $a^* = y$. In this case, by CCC, $a = x$ or $a = z$. If $a = x$, that is, $C(\{x, y, z\}) = \{x\}$, then, by CCC, it would follow that $C(\{x, y\}) = \{x\}$, a contradition. If $a = z$, then, by CRN, noting that $\{y\} \neq C(\{x, y, z\}$ and $\{y\} \neq C(\{y, z\})$, it would follow that $C(\{y, x\}) \neq \{y\}$, a contradiction.

(iii) $a^* = z$. In this case, by CCC, $a = x$ or $a = y$. If $a = x$, then, by CRN, noting that $\{z\} \neq C(\{x, y, z\})$ and $\{z\} \neq C(\{x, z\})$, it would follow that $\{z\} \neq C(\{y, z\})$, a contradiction. If $a = y$, then, by CCC, it would follow that $C(\{y, z\}) = C(\{x, y, z\}) = y$, a contradiction.

In each of the cases (i), (ii), (iii), we would derive a contradiction. Therefore, it must be true that xPz. We have thus shown that the binary relation R is a linear order.

Let the linear order R over X be defined as above. For any non-empty $A \in \mathcal{K}$, let $A = \{a_1, \ldots, a_m\}$ be defined as in Section 2. We show that, for every $A \in \mathcal{K}$, $C(A) = G(A \setminus N(A), R)$, where $N(A) = \{a_1\}$. Consider first that $\#A = 1$. By ESCS, $C(A) = \emptyset$. On the other hand, $N(A) = A = \{a_1\}$ implying $A \setminus N(A) = \emptyset$. Therefore $G(A \setminus N(A), R) = \emptyset$, which implies that $C(A) = G(A \setminus N(A), R)$ for all $A \in \mathcal{K}$ with $\#A = 1$. Next, if $\#A = 2$, from the definition of R, we obtain that $C(A) = \{a_2\} = G(A \setminus N(A), R)$ where $N(A) = \{a_1\}$. Consider therefore $\#A \geq 3$. We first note that a_1 is such that $\{a_1\} \neq C(A)$ and $C(A) = C(A_1)$ for any $A_1 \subseteq A$ with $a_1 \in A_1$ and $C(A) \subseteq A_1$. For, otherwise, for some $a^* \in A$ with $a^* \neq a_1$, by CCC, we would have $\{a^*\} \neq C(A) = C(\{a^*\} \cup C(A))$; by CRN, it would then follow that $\{a^*\} \neq C(\{a^*, a_1\})$, a contradiction with $C(\{a^*, a_1\}) = \{a^*\}$. Let $C(A) = \{a\}$. Note that $a \neq a_1$. If $a \neq a_2$, then, by CCC and from above, we would have $C(\{a, a_1, a_2\}) = \{a\}$. Note that, on the other hand, $\{a\} = C(\{a, a_1\})$ and $\{a\} = C(\{a, a_2\})$. By AU, it would follow that $\{a\} \neq C(\{a, a_1, a_2\})$, a contradiction. Therefore, $C(A) = \{a_2\} = G(A \setminus N(A), R)$, where $N(A) = \{a_1\}$.

In summary, we have shown that if C satisfies ESCS, NENCS, CCC, AU and CRN then it is modesty-norm-constrained rationalizable. ■

Theorem 2. A choice function C is *median-norm-constrained rationalizable* if and only if it satisfiefs axioms NESCS, NENCS, AU, IRA, and MCR.

Proof. It can be checked that if a choice function C is median-norm-constrained rationalizable, then it satisfies axioms NESCS, NENCS, AU, IRA, and MCR. Therefore, we have only to show that if a choice function C satisfies NESCS, NENCS, AU, IRA, and MCR, then it is median-norm-constrained rationalizable.

Let C be a choice function that satisfies NESCS, NENCS, AU, IRA, and MCR. We first note that, by NESCS and NENCS, for all $A \in \mathcal{K}$ with $A \neq \emptyset$, $C(A) \neq \emptyset$. Define the binary relation R over X as follows: for all $x, y \in X$, xPy iff $\{y\} = C(\{x, y\})$, and xIy iff $x = y$. Clearly, R is reflexive, complete and anti-symmetric. To show that R is transitive, we consider distinct $x, y, z \in X$ such that xPy and yPz; that is, $\{y\} = C(\{x, y\})$ and $\{z\} = C(\{y, z\})$. We need to show that xPz, or $\{z\} = C(\{x, z\})$. Suppose to the contrary that $\{z\} \neq C(\{x, z\})$, that is, $\{x\} = C(\{x, z\})$. Consider $C(\{x, y, z\})$. If $C(\{x, y, z\}) = \{x\}$, then, by MCR and noting that $\{z\} \neq C(\{x, y, z\})$ and $\{z\} \neq C(\{x, z\})$, it would follow that $\{z\} \neq C(\{y, z\})$, a contradiction. If $C(\{x, y, z\}) = \{y\}$, then, by MCR and noting that $\{x\} \neq C(\{x, y, z\})$ and $\{x\} \neq C(\{x, y\})$, it would follow that $\{x\} \neq C(\{x, z\})$, a contradiction. If $C(\{x, y, z\}) = \{z\}$, then, by MCR and noting that $\{x\} \neq C(\{x, y, z\})$ and $\{x\} \neq C(\{x, y\})$, it would follow that $\{x\} \neq C(\{x, z\})$, a contradiction. Therefore, $C(\{x, y, z\}) = \emptyset$, a contradiction.

Consequently, $C(\{x,z\}) = \{z\}$ showing that R is transitive. Therefore, R is a linear order.

We now show that $C(A) = G(A \setminus N(A), R)$ for all $A \in \mathcal{K}$ where $N(A) = N_{median}(A)$. From the definition of R, it is trivial to check that $C(A) = G(A \setminus N_{median}(A), R)$ for all $A \in \mathcal{K}$ with $\#A \leq 2$. Consider $A \in \mathcal{K}$ with $\#A = 3$. Let $A = \{x, y, z\}$ with $xPyPz$. From $xPyPz$, we have $\{y\} = C(\{x,y\}), \{z\} = C(\{y,z\})$ and $\{z\} = C(\{x,z\})$. AU implies that $\{z\} \neq C(\{x,y,z\})$. If $C(\{x,y,z\}) = \{x\}$, by MCR and noting that $\{y\} \neq C(\{x,y,z\})$ and $\{y\} \neq C(\{y,z\})$, we would obtain $\{y\} \neq C(\{x,y\})$, a contradiction. Therefore, $C(\{x,y,z\}) = \{y\} = G(\{x,y,z\} \setminus N_{median}(\{x,y,z\}), R)$. Now, for any $A \in \mathcal{K}$ with $\#A \geq 4$, by IRA, there exist distinct $x, y \in A$ such that $C(A) \cap \{x,y\} = \emptyset$ and $C(B) = C(B \setminus \{x,y\})$ for all $B \subseteq A$. Note that $C(\{x,y,a\}) = \{a\}$ for all $a \in A \setminus \{x,y\}$. From the above, it must be true that $\{x,y\} = \{a_1, a_{\#A}\}$. Without loss of generality, let $x = a_1$ and $y = a_{\#A}$. Now, by the repeated use of IRA, it can be checked that $C(A) = G(A \setminus N_{median}(A), R)$.

In summary, we have shown that C is median-norm-constrained rationalizable. ■

Theorem 3. A choice function C is *protest-norm-constrained* rationalizable if and only if it satisfies axioms SPC, NENPS and RC.

Proof. It can be checked that if a choice function C is protest-norm-constrained rationalizable, then it satisfies axioms SPC, NENPS and RC. Therefore, we have only to show that if a choice function C satisfies SPC, NENPS and RC, then it is protest-norm-constrained rationalizable.

Let C be a choice function that satisfies SPC, NENPS and RC. Let $X_1 = \{x \in X : C(\{x\}) = \emptyset\}$. If $X_1 = X$, then define, for each $A \in \mathcal{K}$, $N(A) = A$. Note that in this case we have $C(\{x\}) = \emptyset$ for all $x \in X$. Then, for any linear order R, we have $G(A \setminus N(A), R) = \emptyset$. On the other hand, from $C(\{x\}) = \emptyset$ for all $x \in X$, by SPC, we obtain that, for all $A \in \mathcal{K}$, $C(A) = \emptyset$. Therefore, $C(A) = G(A \setminus N(A), R)$ for any linear order R. If $X_1 \neq X$, then the set $X_2 = X \setminus X_1 = \{x \in X : C(\{x\}) = \{x\}\} \neq \emptyset$. For each $A \in \mathcal{K}$, let $N(A) = \{a \in A : a \in X_1\}$. Let $X_1 = \{x_{11}, \ldots, x_{1p}\}$. Define the binary relation R over X as follows: for all $x, y \in X$,

xIy iff $x = y$;

xPy if $[x, y \in X_2$ and $\{x\} = C(\{x,y\})]$ or $[x \in X_2$ and $y \in X_1]$ or $[x, y \in X_1,\ x = x_{1i}, y = x_{1j}$ and $i > j]$.

From the definition of R, R is reflexive, complete and anti-symmetric. We now check that R is transitive. Let distinct $x, y, z \in X$ be such that xPy and yPz. We need to show that xPz. Suppose to the contrary that not(xPz). Then, we would have zPx. There are three cases: (i) $x, z \in X_2$ and $\{z\} = C(\{x,z\})$; (ii) $z \in X_2$ and $x \in X_1$; and (iii) $x, z \in X_1$ with $z = x_{1i}$, $x = x_{1j}$ and $i > j$. In (i), from $z \in X_2$ and $\{y\} = C(\{y,z\})$, by SPC, we would obtain $C(\{y\}) \neq \emptyset$. Therefore, $x, y, z \in X_2$ in this case. Consider $C(\{x,y,z\})$. It can be checked that, by RC, any of the following, $C(\{x,y,z\}) = \{x\}$, $C(\{x,y,z\}) = \{y\}$, and $C(\{x,y,z\}) = \{z\}$, would lead a contradiction. As a consequence, in this case, we would have $C(\{x, y, z\}) = \emptyset$, a contradiction with NENPS. In case (ii), from $z \in X_2$ and $\{y\} = C(\{y, z\})$, by SPC, we would obtain $C(\{y\}) \neq \emptyset$, that is, $y \in X_2$, leading an immediate contradiction with xPy and $x \in X_1$. Finally, in case (iii), $x \in X_1$ and xPy would imply that $y \in X_1$ with $y = x_{1k}$ and $j > k$, and yPz would then imply $k > i$, an immediate contradiction with $i > j$. Therefore, we must have xPz, proving the transitivity of the binary relation R, and the linearity of R.

To complete the proof of Theorem 3, we show that, for all $A \in \mathcal{K}$, $C(A) = G(A \setminus N_{protest}(A), R)$. Let $A \in \mathcal{K}$. We first observe that, for any $x \in N_{protest}(A)$, by SPC, $x \notin C(A)$. If $C(A) = \emptyset$, by SPC and NENPS, it must be the case that $N_{protest}(A) = A$. Then, $G(A \setminus N_{protest}(A), R) = \emptyset$. Similarly, when $G(A \setminus N_{protest}(A), R) = \emptyset$, indicating that $N_{protest}(A) = A$, by SPC, $C(A) = \emptyset$ follows easily. If $C(A) = \{a\}$, then $\{a\} = C(\{a\})$. Then, $N_{protest}(A) \neq A$, and $G(A \setminus N_{protest}(A), R) \neq \emptyset$. If $\{x\} = G(A \setminus N_{protest}(A), R)$ and $x \neq a$, by RC and noting that $C(A) = \{a\}$ and $\{a\} = C(\{a\})$, we would have $C(\{x, a\}) = \{a\}$, implying that aPx, a contradiction to $\{x\} = G(A \setminus N_{protest}(A), R)$ and $x \neq a$. Therefore, $a = x$. Similarly, when $\{x\} = G(A \setminus N_{protest}(A), R)$, it must be true that $\{x\} = C(\{x\})$. If $C(A) = \{a\}$ and $a \neq x$, by SPC, we would have $\{a\} = C(\{a\})$, and by RC, we would then have $C(\{x, a\}) = \{a\}$, a contradiction with $\{x\} = G(A \setminus N_{protest}(A), R)$ and $a \neq x$. Therefore, $a = x$. In sum, we have established that, for all $A \in \mathcal{K}$, $C(A) = G(A \setminus N_{protest}(A), R)$, completing the proof of Theorem 3. ∎

Theorem 4. A choice function C is *weakly protest-norm-constrained* rationalizable if and only if it satisfies axioms WPC, NENPS and RC.

Proof. It can be checked that if a choice function C is weakly protest-norm-constrained rationalizable, then it satisfies axioms WPC, NENPS and RC. Therefore, we have only to show that if a choice function C satisfies WPC, NENPS and RC, then it is weakly protest-norm-constrained rationalizable.

Let C be a choice function that satisfies WPC, NENPS and RC. If $C(\{x\}) = \emptyset$ for all $x \in X$, then define the norm $N(A) = A$ for all $A \in \mathcal{K}$, and for any linear order R, we must have $C(A) = G(A \setminus N(A), R)$ since, by the repeated use of WPC, $C(A) = A$ for all $A \in \mathcal{K}$. Therefore, let X be such that $C(\{x\}) = \{x\}$ for at least one $x \in X$. By NENPS, it then follows that $C(X) \neq \emptyset$. Let $C(X) = \{x_1\}$. Consider the following subsets of X:

$X_1 = X \setminus \{x_1\}$,
$X_2 = X_1 \setminus C(X_1)$ if $C(X_1) \neq \emptyset$ and $X_2 = X_1$ if $C(X_1) = \emptyset$,
$X_3 = X_2 \setminus C(X_2)$ if $C(X_2) \neq \emptyset$ and $X_3 = X_2$ if $C(X_2) = \emptyset$,
...
$X_{\#A} = X_{\#A-1} \setminus C(X_{\#A-1})$ if $C(X_{\#A-1}) \neq \emptyset$ and $X_{\#A} = X_{\#A-1}$ if $C(X_{\#A-1}) = \emptyset$.

We note that, if, for some k, $C(X_k) \neq \emptyset$ and $C(X_{k+1}) = \emptyset$, then $X_j = X_{k+1}$ for all $j \geq k + 1$. Let k^0 be such that $C(X_{k^0}) = \emptyset$ and $C(X_{k^0-1}) \neq \emptyset$. If $X_{k^0} \neq \emptyset$, let $X_{k^0} = \{a_1, \ldots, a_m\}$. Define a binary relation R over X as follows: for any distinct $x, y \in X$,

$$xIx,$$

xPy if $[\{x\} = C(X_i),\ \{y\} = C(X_j)$ and $i < j]$ or $[\{x\} = C(X_i),$ $y \in X_j$ and $C(X_j) = \emptyset]$ or $[x, y \in X_{k^0},\ x = a_i, y = a_j$, and $i < j]$.

From the definition, it can be checked that the binary relation R is reflexive, complete, anti-symmetric and transitive. We now show that, for any $A \in \mathcal{K}$, $C(A) = G(A \setminus N_{w.protest}(A), R)$. Let $A \in \mathcal{K}$. If A is such that $C(\{x\}) = \emptyset$ for all $x \in A$, then, by WPC, $C(A) = \emptyset$. Note that $N_{w.protest}(A) = A$. On the other hand, $G(A \setminus N_{w.protest}(A), R) = G(\emptyset, R)$. Therefore, in this case, $C(A) = G(A \setminus N_{w.protest}(A), R)$. If A is such that, for some $x \in A$, $C(\{x\}) = \{x\}$, then $C(A) \neq \emptyset$ and $N_{w.protest}(A) = \emptyset$. Let $\{a\} = C(A)$, and $G(A \setminus N_{w.protest}(A), R) = \{x\}$. We need to show that $a = x$. Suppose to the contrary that $x \neq a$. Then, since $G(A \setminus N_{w.protest}(A), R) = G(A, R) = \{x\}$ and $a \in A$, we must have xPa. From the definition of R, from xPa, we must have either (i) $\{x\} = C(X_i), \{a\} = C(X_j)$ and $i < j$; or (ii) $\{x\} = C(X_i), a \in X_j$ and $C(X_j) = \emptyset$; or (iii) $x, a \in X_{k^0}, x = a_i, a = a_j$, and $i < j$. In (i), we first note that $X_j \subset X_i$ and $X_i \neq X_j$. If $A \subseteq X_i$, then, by noting that $C(A) \neq \emptyset$ and from RC, we obtain $\{x\} = C(A)$, a contradiction. If $X_i \subseteq A$, then by noting that $C(X_i) \neq \emptyset$ and from RC, we obtain $C(X_i) = \{a\}$, a contradiction. If A is not subset of X_i and X_i is not a subset of A, then $D = (A \setminus X_i) \cup (X_i \setminus A) \neq \emptyset$. For each $y \in D$, from the construction of $X_1, X_2, \ldots$, it must be the case that $\{y\} = C(X_{k(y)})$ for some $k(y)$ with $k(y) < i$. Let $k(y^*)$ be the smallest integer among those $k(y)$. Note that $A \subseteq X_{k(y^*)}$ and $X_i \subset X_{k(y^*)}$. By RC, it then follows that $\{y^*\} = C(A)$. If $y^* = a$ implying that aPx, we obtain the contradiction with xPa; if $y^* \neq a$, then, by RC, it follows that $\{y^*\} = C(A)$, a contradiction with $C(A) = \{a\}$. Therefore, case (i) is not possible. In case (ii), again, when $A \subseteq X_i$ or $X_i \subseteq A$, we obtain immediate contradictions. Therefore, consider $D = (A \setminus X_i) \cup (X_i \setminus A) \neq \emptyset$. Then, by following a similar argument as in case (i), we obtain a contradiction. Therefore, case (ii) is not possible. Finally, in case (iii), since $C(A) \neq \emptyset$, we must have $y \in A$ with $C(\{y\}) \neq \emptyset$. Let $k(y^*) \in A$ be such that $\{y^*\} = C(X_{k(y^*)})$ and $A \subseteq X_{k(y^*)}$. From the construction, $k(y^*) < i$, so that $y^* \neq x$ and y^*Px, a contradiction. Therefore, case (iii) is not possible. On the other hand, the above three cases exhaust all possibilities. Therefore, $x = a$, completing the proof of Theorem 4. ■

4. Conclusion

In this paper, we have proposed a general and unified framework to study norm-constrained choice behaviors. In this framework, we have introduced a general notion of norm-constrained rationalizability of a choice function. Our notion of rationalizability is general enough to include the conventional notion of rationalizability of a choice function as a special case in which the norm is characterized by $N(A) = \emptyset$ for all non-empty feasible set A. With our notion of rationalizability of a choice function, we have studied several concepts of specifically norm-constrained choices axiomatically.

To conclude the paper, we note that there have been some attempts in the literature recently on *sequential* rationalizability of a choice function. For example, Manzini and Mariotti (2005) introduce a notion of rationalizability of the following type to study cyclic choice behaviors: a choice function C is rationalizable if there exist two orderings (an ordering is a reflexive, complete and transitive binary relation), R_1 and R_2, such that $C(A) = G(G(A, R_1), R_2)$ for all non-empty $A \in \mathcal{K}$. Clearly, this notion of rationalizability is quite different from ours. The motivation of their study is also quite different from ours.

Bibliography

Aizerman, M./F. Aleskerov (1995), *Theory of Choice,* Amsterdam

Baigent, N./W. Gaertner (1996), Never Choose the Uniquely Largest: A Characterization, in: *Economic Theory 8*, 239–249

Gaertner, W./Y. Xu (1997), Optimization and External Reference: A Comparison of Three Linear Axiomatic Systems, in: *Economics Letters 57*, 57–62

—/— (1999a), On the Structure of Choice Under Different External References, in: *Economic Theory 14*, 609–620

—/— (1999b), On Rationalizability of Choice Functions: A Characterization of the Median, in: *Social Choice and Welfare 16(4)*, 629–638

—/— (1999c), Rationality and External Reference, in: *Rationality and Society 11*, 169–185

—/— (2004), Procedural Choice, in: *Economic Theory 24(2)*, 335–349

Hansson, S. O. (1996), Social Choice with Procedural Preferences, in: *Social Choice and Welfare 13*, 215–230

Manzini, P./M. Mariotti (2005), *Rationalizing Boundedly Rational Choice*, Mimeo, Queen Mary College, University of London

Rabin, M. (1993), Incorporating Fairness into Game Theory and Economics, in: *American Economic Review 83*, 1281–1302

Sen, A. K. (1977), Social Choice Theory: A Re-examination, in: *Econometrica 45*, 53–89

— (1988), Freedom of Choice: Concept and Content, in: *European Economic Review 32*, 269–294

— (1933), Internal Consistency of Choice, in: *Econometrica 61*, 495–521

Suzumura, K. (1999), Consequences, Opportunities, and Procedures, in: *Social Choice and Welfare 16*, 17–40

—/Y. Xu (2001), Characterizations of Consequentialism and Non-consequentialism, in: *Journal of Economic Theory 101*, 423–436

Suzumura, K./Y. Xu (2003), Consequences, Opportunities, and Generalized Consequentialism and Non-consequentialism, in: *Journal of Economic Theory 111*, 293–304

—/— (2007), *Consequentialism and Non-consequentialism: The Axiomatic Approach*, Mimeo, Department of Economics, Georgia State University