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Instances of Indeterminacy*

Abstract: This paper is a survey of how economists and philosophers approach the issue of comparisons. More precisely, it is about what formal representation is appropriate whenever our ability to compare things breaks down. We restrict our attention to failures that arise with ordinal comparisons. We consider a number of formal approaches to this problem including one based on the idea of parity. We also consider the claim that the failure to compare things is a consequence of vagueness. We contrast two theories of vagueness; fuzzy set theory and supervaluation theory. Some applications of these theories are described.

0. Introduction

Imagine that a university department is in the process of appointing a new member, in order to fill a vacant position. Five candidates have been interviewed for the position and each member of the department has been asked to rank them. Imagine that a member of this department is comparing two of the candidates, and what he or she cares about is how they fare with respect to teaching and research. Imagine that one of the candidates (candidate A) is better at research than the other (candidate B). However, just to complicate matters, imagine that candidate A is worse at teaching than candidate B. How would our department member place these two candidates in order? Often it is hard to say, but not always.

For instance, imagine that candidate A is much better at research than candidate B and only slightly worse at teaching. In such cases, it seems reasonable to suppose that our department member would place A above B in his or her ranking. The reason for this is that most members of a university department would probably be willing to trade-off slightly inferior teaching quality in order to acquire a colleague who is significantly better at research.

Unfortunately, things are not always as straightforward as this. For instance, what if candidate A is not only worse at teaching than candidate B but significantly worse? In cases like this, it might be extremely difficult for the department member to place the two candidates in a clear order. On the one hand, he or

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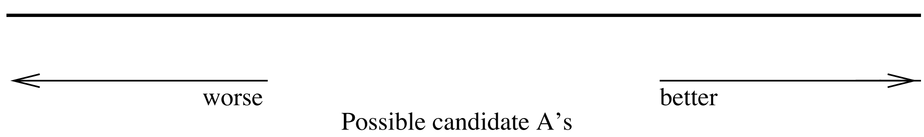
she might feel that to some extent candidate A is better than candidate B. At the same time however, he or she might also feel that to some extent candidate B is better than candidate A. These conflicting feelings may be difficult to integrate into a clear expression of preference or indifference. We call cases like this "instances of indeterminacy".

Here is another example, which is based on an argument in Sen (1970). Imagine that a social planner is attempting to rank two social states, X and Y, in which only two people live. Suppose that person i is better-off in X than in Y, but that person j is better-off in Y than in X. In other words, X and Y cannot be ranked by the traditional Pareto principle. Does this mean that the planner cannot put the two states in order? Quite correctly, Sen argues no. For example, it might be the case that state X is much better for person i than state Y, and yet only slightly worse for person j. If so, it would be reasonable for the planner to place X above Y in the social ordering. In other words, the planner could judge that the gains to person i in state X outweigh the losses to person j.

However, as Sen points out, this does not mean that it is always possible to rank Pareto incomparable states. For example, if the gains to person i in X are similar to the losses to person j, then it might be extremely difficult for the planner to place the two social states in a clear order.[1] The planner might feel that to some extent state X is better than state Y. At the same time however, the planner might also feel that to some extent state Y is better than state X. Just like in our previous example, these conflicting feelings may be difficult to integrate into a clear expression of preference or indifference. This is another instance of indeterminacy. In the first example the indeterminacy is in an individual's preference relation whereas in the second example it is in a social preference (or "betterness") relation.

1. Standard Configurations

A useful device for thinking about these examples is what Broome (1997) calls a "standard configuration". Take our original example of the two job candidates, A and B. Imagine that the quality of candidate A's research is fixed but that the quality of his teaching can vary continuously. Possible candidate A's form a "chain" which Broome (1997) represents by points on a continuous line like this:



Each point on the line represents a possible candidate A. Each possible candidate A on the line is better than every candidate to the left of it. This

[1] Sen describes this as a model of "partial" comparability of welfare units. Sen's partial comparability model has not really been developed in the large literature on interpersonal comparisons, as is clear from the survey articles of d'Aspremont and Gevers 2002 and Bossert and Weymark 2004. Exceptions to this include Blackorby 1975 and Basu 1980.

is because, as we move from left to right, the quality of A's teaching increases. A standard configuration consists of a continuous chain like this one, together with single alternative known as the "standard". In our example, the standard is candidate B. Candidate B's teaching and research quality are held constant. An important feature of Broome's configuration is that the chain must be sufficiently long enough to ensure that the worst candidate A's are worse than candidate B and the best candidate A's are better than candidate B.

As should be clear, standard configurations can be drawn up in other contexts. Broome gives the example of churches. We could imagine a chain of churches, each less impressive than the next. At the left of this chain would be some unimpressive chapel, but at the right would be something magnificent like St Peter's. In Broome's example, the standard is Stonehenge. The churches at the left of the chain are less impressive than Stonehenge, but the ones at the right are more impressive.

With the standard configuration in place, we can compare options in the chain with the standard - in our example, possible candidate A's against candidate B. At the left there are possible candidate A's who are worse than candidate B; they form the "worse zone". At the right are possible candidate A's who are better than candidate B; they form the "better zone". Broome is interested in what happens in between these zones.

Broome argues that as we move along the chain from left to right, we often encounter what he calls a "zone of indeterminacy". He says the following, referring to his example of churches:

> I think St Peter's is more impressive than Stonehenge, and Stonehenge is more impressive than the gospel chapel in Stoke Pewsey, but for some churches in the chain the comparison with Stonehenge will be indeterminate. (Broome 1997, 69).

Broome gives Salisbury Cathedral as an example of a church which falls within the zone of indeterminacy for this configuration. He writes:

> Which is more impressive: Salisbury Cathedral or Stonehenge? I think there is no determinate answer to this question. The dyadic predicate 'more impressive than' - the comparative of the monadic predicate 'impressive' - seems to allow indeterminate cases. Many comparatives are like that. Amongst them are many evaluative comparatives, such as 'lovelier than', 'cleverer than', and the generic 'better than'. (Broome 1997, 67).

Broome conjectures that the width of the zone of indeterminacy depends on how similar the alternatives in the chain are to the standard. Since Stonehenge and churches are dissimilar, the zone of indeterminacy in that configuration is probably quite wide.[2] However, in the job candidates example, the standard is sufficiently similar to the alternatives in the chain that the zone of indeterminacy is probably quite narrow.

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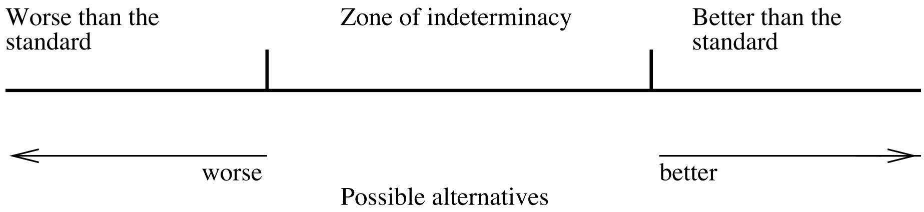
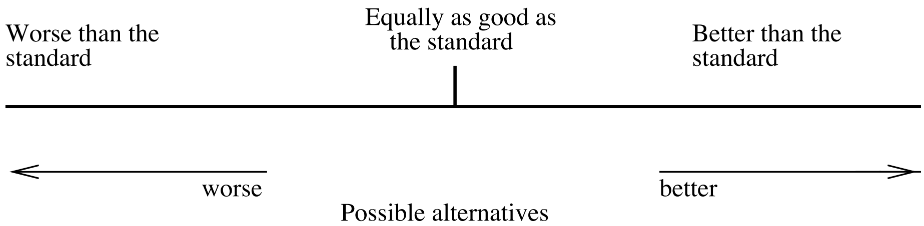
[2] According to the most recent theory, Stonehenge was a hospital.

We can represent the zone of indeterminacy in a picture, like the one given earlier. It is again taken from Broome:



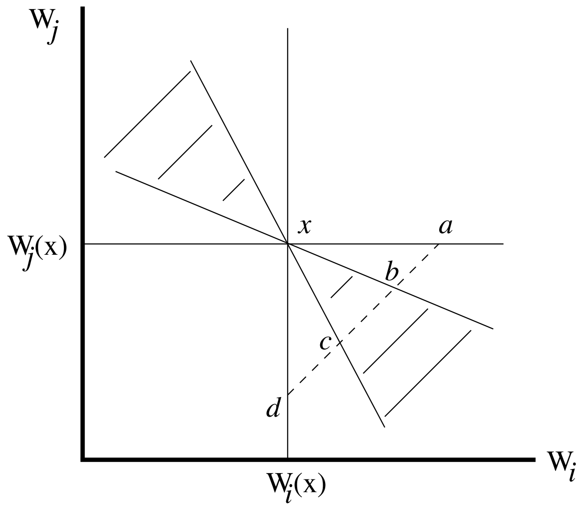
It is important to emphasise, as Broome does, that no alternative within the indeterminate zone can be equally as good as the standard. Imagine that Salisbury Cathedral is equally as impressive as Stonehenge. Then anything that is more impressive than Salisbury Cathedral is more impressive than Stonehenge. Equally, anything that is less impressive than Salisbury Cathedral is less impressive than Stonehenge.[3] But if this is the case then the zone of indeterminacy does not exist. Instead we have the following picture:



In this configuration, one alternative in the chain is equally as good as the standard and every other is either in the better zone or the worse zone. In other words, this is a configuration where there is no indeterminacy. It is implausible that all configurations have this property.

Configurations with a zone of indeterminacy can arise in welfare economics. A theme in work by Sen (1992) is that an individual's well-being is determined in part by their capability to achieve certain functionings. Health and literacy are examples of functionings. Raising a person's capability to achieve more of these functionings increases their well-being, although Sen argues that it is often unclear as to how functionings should be traded off against one another. He proposes that a range of weights be used, each weight establishing a particular trade-off. The following figure is adapted from Sen (1992, 47).

[3] This reasoning appeals to an extended transitivity principle. This principle says that for any x, y, and z, if x is Fer than y and y is equally as F as z, or if x is equally as F as y and y is Fer than z, then x is Fer than z. Note that "Fer than" refers to some comparative like "more impressive than". Broome takes this extended transitivity condition to be a principle of logic. Whilst this is true of comparatives, whether or not preferences are comparatives is an open question. If not, then there is no reason to suppose that extended transitivity will hold for preferences. Luce 1956 is the classic reference. Broome 1991 argues that when we interpret a social preference relation as a betterness relation then it must be transitive since "better than" is the comparative of the property "good".



In this figure we assume that an individual's level of functioning i can be measured by a real number W_i. Similarly, his or her level of functioning j is measured by W_j. The individual's current "state of being" is, therefore, represented by the vector $(W_i(x), W_j(x))$. Each weight determines a line through x, running from top-left to bottom-right. These lines are the familiar "indifference curves" from economics and for simplicity we assume that they are linear. As these weights are varied within the appropriate range, we trace out two shaded cones, one north-west of x, the other south-east of x. With these cones in place, we can construct a continuous chain of functionings from a to d. Functionings between a and b are better for individual i than x, and functionings between c and d are worse for individual i than x. The part of the chain from b to c forms a zone of indeterminacy.

At this stage it is worthwhile setting up a definition of "indeterminate" that we will use throughout this paper. A natural definition is this one: it is indeterminate as to which of x and y is at least as good as the other if and only if it is not true that x is better than y, not true that y is better than x, and not true that x and y are equally good. This definition turns out to be fairly neutral, although it is not uncontroversial as we explain later on.

2. Representations

2.1 Incomplete Orderings

The traditional approach in economics (and philosophy) is to represent indeterminate comparatives by incomplete orderings. Let R be a binary relation on X, so writing xRy means that x is at least as good as y. R is assumed to be reflexive and transitive.[4] In addition, R is complete if and only if for all distinct $x, y \in X$, $xRy \vee yRx$. This disjunction is intended in an inclusive sense. From this primitive relation R, preference and indifference can be defined. Typically preference is represented by a relation P and is defined as xPy if and only if $xRy \wedge \neg yRx$; indifference is represented by a relation I and is defined as xIy if and only if $xRy \wedge yRx$.[5]

A reflexive and transitive binary relation is called a quasi-ordering (or a preorder). A complete quasi-ordering is called an ordering. An incomplete ordering is a quasi-ordering for which the completeness assumption fails to hold. That is, there exists a pair of alternatives $x, y \in X$, such that $\neg(xRy \vee yRx)$ which is equivalent to $\neg xRy \wedge \neg yRx$. A relation N of "incomparability" can be defined as xNy if and only if $\neg xRy \wedge \neg yRx$. N is symmetric (xNy implies yNx) but not, in general, transitive. It is easy to see that xNy is equivalent to $\neg xPy \wedge \neg yPx \wedge \neg xIy$.

In words, what this means is as follows. If we take some alternative in the zone of indeterminacy and compare it with the standard, then it is false that it is better than the standard, false that the standard is better than it, and false that it and the standard are equally good. Broome (1997) refers to this as "hard indeterminacy". It is consistent with the definition of indeterminacy we gave earlier; a statement that is false is not true.

One argument in favour of the traditional approach comes from the principle of bivalence. The principle of bivalence says that for any proposition "P", either "P" is true or "P" is false. Given our definition, if the principle of bivalence is true then hard indeterminacy is the only possible way of representing the indeterminacy in the comparative "better than".

However, many philosophers do not believe in the principle of bivalence. Typically these philosophers explain this fact by appealing to the existence of vague predicates. Natural language, it is claimed, is infected with vague predicates. Predicates such as "thin", "tall", "red" and "impressive" are often claimed to be vague. These predicates are vague because they admit borderline cases. For example, some people are borderline thin. Some reddish-orange colour patches are borderline red, and so on. Vagueness is the phenomenon of borderline cases.

According to the standard view in philosophy, vagueness creates problems for classical logic.[6] For example, if Jim is borderline thin then the proposition "Jim is thin" is neither true nor false, violating the principle of bivalence. The law of excluded middle, that "Jim is thin or Jim is not thin" is true, seems suspect too.

Philosophers were first moved to study vagueness because of the so-called sorites paradox. Here is one version of it.

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[4] A relation R on X is reflexive if and only if, for all $x \in X$, xRx. R is transitive if and only if, for all $x, y, z \in X$, $xRy \wedge yRz \rightarrow xRz$.

[5] Extended transitivity can be defined in this context by: for all $x, y, z \in X$, $xPy \wedge yIz \rightarrow xPz$. It is easy to show that, if R is complete, R is transitive if and only if P is transitive and extended transitivity is satisfied.

[6] The standard view has been challenged by Williamson 1994. Keefe and Smith 1997, 1–57, is an excellent introduction to the philosophical literature on vagueness. They describe several competing theories of vagueness. Keefe and Smith 1997 is an anthology. Another non-overlapping anthology is Graff and Williamson 2002. Keefe 2000 is also recommended.

Consider the following two premises.

1. For all n, where n is a number, if "n grains of sand is not a heap" is true then "$n + 1$ grains of sand is not a heap" is also true.

This is a conditional of the form "if p then q". The second premise is uncontroversial.

2. "1 grain of sand is not a heap" is true.

Accepting these premises and invoking the logical rule *modus ponens*, we can derive the conclusion that "10,000 grains of sand is not a heap" is true. However, it is false. This is the paradox.

If the principle of bivalence is false, then the traditional account of the indeterminacy in "better than" is not forced on us.

2.2 Parity

There is a literature in philosophy which argues that two alternatives can be compared even when it is not true that one is better than the other, and not true that they are equally good. This is how we defined indeterminacy earlier in the paper. According to this literature, that definition is wrong. In its place should be something like this: it is indeterminate as to which of x and y is at least as good as the other if and only if it is not true that x is better than y, not true that y is better than x, not true that x and y are equally good, and not true that x and y are *on a par*. These philosophers think that there is a distinct value relation "on a par" which exists in addition to "better than" and "equally good".

Perhaps the two philosophers most closely associated with this view are Griffin (1986; 1997) and Chang (1997; 2002), although similar ideas can be found in unpublished work by Derek Parfit.[7] Griffin talks of "rough equality" whereas Chang uses "on a par".

An important formalisation of this approach within a bivalent framework is given by Qizilbash (2002; 2005). Qizilbash accepts all of the definitions of the binary relations we introduced in the last section, but augments them with a new one. This new binary relation C is defined on X with xCy meaning that x is comparable with y. Qizilbash's theory has, therefore, two binary relations as primitives, R and C, not one. C is reflexive and symmetric. He assumes that $(xRy \vee yRx)$ implies xCy but not the converse. New binary relations are defined from these primitives. Parity is represented by a relation O with xOy meaning that x is on a par with y; this is defined as xOy if and only if $xCy \wedge xNy$. Indeterminacy (or what Qizilbash calls "incomparability") is represented by a relation D; this is defined as xDy if and only if $\neg xCy$.

An alternative formalism is given by Rabinowicz (2007). Rabinowicz suggests that we consider a non-empty set K of transitive and reflexive binary relations

[7] We do not know whether these philosophers accept the principle of bivalence. The definition of indeterminacy that we attribute to them is neutral in this respect.

that represent preferences. He does not require that these binary relations are complete. He proposes that we take the intersection of all of these binary relations: x is at least as good as y if and only if x is at least as good as y in every binary relation in K, x and y are fully comparable if in every binary relation in K, either x is at least as good as y or y is at least as good as x, x and y are on a par if K contains two binary relations such that x is preferred to y in one binary relation and y is preferred to x in the other. They are fully on a par if, in addition to being on a par, they are fully comparable. Finally, x and y are incomparable if for every binary relation $W \in K$ we have $(x, y), (y, x) \notin W$. As Rabinowicz notes, the idea of intersection modelling is old, going back at least to Sen (1973).

It is clear from both of these contributions that parity can be defined in a formal way. But does this extra value relation exist? Here we focus on Chang's (2002) argument. Imagine that we take an alternative in the zone of indeterminacy and one in the worse zone. The alternative in the worse zone is worse than the standard. It is also worse than the alternative in the zone of indeterminacy. We can now imagine a sequence of items in the chain, starting with the alternative in the worse zone and ending with the alternative in the zone of indeterminacy. In this sequence each successive alternative is slightly better than its immediate predecessor. Of course, at some point in this sequence we cross over into the zone of indeterminacy. In other words, there will be an alternative in this sequence that is comparable with the standard, but its immediate successor isn't. Chang argues that a small increase in the value of an alternative cannot make comparability disappear whenever it was present initially (she calls this the "chaining argument"). For Chang, comparability does not disappear when we enter the zone. The zone of indeterminacy is simply a zone of parity.

Chang's argument relies on an attractive intuition that is also present when we consider the sorites paradox. This intuition says that small changes cannot make a difference to the application of certain predicates. However, we know from the sorites paradox that this intuition is false when the predicates in question are vague.[8] If "comparable with the standard" is a vague predicate then Chang's argument is simply a version of the sorites paradox. Chang argues that "comparable" is not vague but, as she admits, her argument is not decisive.

To see why, consider the possibility that the alternative in the worse zone (at the beginning of the sequence) is comparable with the standard, whereas the alternative in the indeterminate zone (at the end of the sequence) is incomparable with the standard. As we move from left to right along the chain of alternatives it is reasonable to think that the boundary of the predicate "comparable with the standard" is vague, not sharp.[9] It is to this possibility that we now turn. A striking conclusion we reach if we accept the following analysis is this: the entire zone of indeterminacy is a zone of vagueness.

[8] The dominant theories of vagueness in philosophy resolve the sorites paradox by rejecting the first premise.

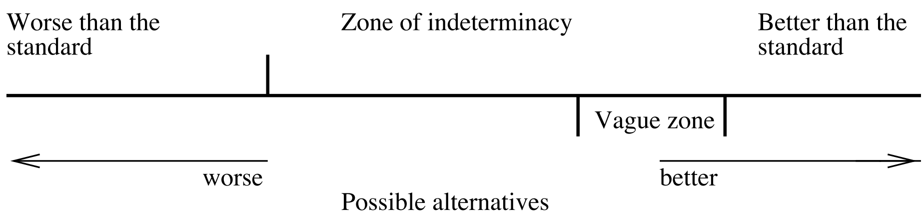
[9] This point is also made by Rabinowicz 2007.

3. Vagueness

One way of criticising hard indeterminacy is the following.[10] Imagine that we move from right to left along the chain of alternatives. That is, we start with alternatives that are better than the standard, and progressively make them worse. Then there is a sharp transition from those that are better than the standard, to those that are not better. This transition occurs as we move from the better zone into the zone of indeterminacy. Intuitively, however, we might think that the boundaries of the predicate "better than the standard" are vague and not sharp. In other words, as we move from right to left along the chain of alternatives, intuition suggests that the truth value of the proposition "This point is better than the standard" does not switch from true to false at some particular point. Here is a quote from Broome. Note that the configurations Broome refers to in this quote run from top to bottom, not right to left.

> Take "redder than", for instance. "Redder than the standard" is a monadic predicate, and it is natural to think it may be vague, like the predicate "red". Start from the top of the chain in the standard configuration I described, which runs from red to yellow. Points at the top of the chain are red, and it seems implausible that as we move down the chain we encounter a sharp boundary that divides these points that are red from those that are not red. Similarly, points at the top are redder than the standard (which is reddish-purple and not in the chain itself), and one might find it implausible that, as we move down the chain, we encounter a sharp boundary that divides these points that are redder than this standard from points that are not redder than it. Instead, it seems that there may be a zone of vagueness between points that are redder than the standard and those that are not. (Broome 1997, 73).

We can illustrate this possibility in the following picture:



The existence of this vague zone seems natural.[11] What can we say about how the alternatives in the vague zone compare with the standard? As we have seen, the traditional view in philosophy is that propositions with vague terms are

[10] This argument is due to Broome 1997.

[11] There will be another vague zone at the left of the zone of indeterminacy, between points that are worse than the standard and those that are not. We have omitted this in order to keep the picture simple.

neither true nor false. So it is not true that an alternative in the vague zone is better than the standard, but it is not false either. Broome argues that the vague zone cannot exist if the zone of indeterminacy is "hard". This is a crucial feature of his analysis and it relies on something he calls the "collapsing principle". A special version of the collapsing principle says that if it is false that y is Fer than x, and not false that x is Fer than y then it is true that x is Fer than y, where "Fer than" is a comparative. Broome takes the collapsing principle to be obvious, but offers a brief argument in its favour nonetheless.[12] Why does this principle imply that hard indeterminacy and the vague zone cannot coexist? Imagine that x is in the vague zone and y is the standard. This means that it is not true that x is better than y, and not false that x is better than y. However, we know for sure that it *is* false that y is better than x. This is false at all points in the zone of indeterminacy, and to the right of it. But if it is false that y is better than x and not false that x is better than y, then the collapsing principle says that it must be true that x is better than y. However, our assumption is that this is not true. Another way of saying this is that the set of propositions {false that y is better than x, not false that x is better than y, not true that x is better than y} is inconsistent under the collapsing principle. Hard indeterminacy and the vague zone cannot coexist.

How is consistency restored? Suppose that we are attracted to the idea of a vague zone, and accept the collapsing principle. This means that it is not true that x is better than y and not false that x is better than y. If y is better than x then it is false that x is better than y.[13] However, our assumption is that this is not false. Therefore, it is not true that y is better than x. Is it false? If so, then the collapsing principle says that it is true that x is better than y, but our assumption is that this is not true. Therefore, it is not false that y is better than x. This means that our two initial propositions and the collapsing principle entail the following set of consistent propositions (not false that x is better than y, not true that x is better than y, not false that y is better than x, not true that y is better than x).

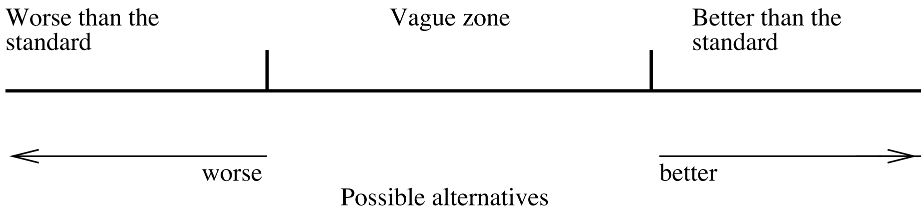
This opens up the possibility of a new kind of indeterminacy which is consistent with the definition we gave at the start of the paper. Broome calls it "soft indeterminacy". This definition says that it is indeterminate as to which of x and y is at least as good as the other if and only if it is neither true nor false that x is better than y and neither true nor false that y is better than x. In addition, it cannot be true that x and y are equally good. As we have seen, if x and y are equally good then the zone of indeterminacy cannot exist.

The striking conclusion of Broome's analysis is that the entire zone of indeterminacy is a zone of vagueness. This means that instances of indeterminacy are just borderline cases of a vague predicate. We can illustrate this in the following picture:

Broome's analysis depends critically on the collapsing principle. The collapsing principle is controversial. Carlson (2004) argues against it. Moreover, one of Broome's original objections to hard indeterminacy was that sharp transitions in

[12] Broome 1997, 74–75.

[13] This is a consequence of the asymmetry of "better than".



truth value occur as we enter the zone of indeterminacy. Of course, a transition still occurs under soft indeterminacy. As we move from the better zone into the zone of indeterminacy, the proposition "This point is better than the standard" switches from true to not true, but it is not false either. There is still a transition then, but Broome regards it as a softer transition.

3.1 Supervaluation Theory

In this section we describe one of the popular theories of the logic of vague terms, namely supervaluation theory. This theory was originally described by Mehlberg (1958) but it was developed more fully under the name "supervaluation" by van Fraassen (1966; 1968). In addition to van Fraassen's work, Dummett (1975) and Fine (1975) are regarded as two of the seminal papers.

Supervaluation theory makes the following claim.

Supervaluation theory: A proposition containing a vague term is true if and only if it is true in all sharpenings of the term.

Intuitively, a sharpening is an admissible way in which a vague term can be made precise. Consider a vague predicate like "old". Each admissible way of making "old" precise is a sharpening of that term. For instance, "old" can be interpreted as being 60 years of age and over. Alternatively, it can be interpreted as being 65 years of age and over, and so on. Of course, being 10 years of age and over cannot be a sharpening of "old" since it is clearly not an admissible way of making that predicate precise. The notion of admissibility is, therefore, a primitive of the theory. By sharpening its terms, any proposition containing vague terms can also be sharpened in a number of different ways.

Supervaluationists reject the principle of bivalence. To illustrate, imagine that "old" can be sharpened in the ways we described. Suppose that Jim is 64 years old. Then the proposition "Jim is old" is true in some sharpenings and false in others. Since "Jim is old" is not true in all admissible sharpenings, then according to supervaluation theory it is not true. Similarly, "Jim is not old" is not true either since it is not true in all admissible sharpenings. A proposition is false if and only if its negation is true. So the proposition "Jim is old" is not false. The proposition "Jim is old" is neither true nor false. The principle of bivalence fails.

An important feature of supervaluation theory, however, is that the law of excluded middle holds. "Jim is old or Jim is not old" is true in all sharpenings and so is true according to supervaluation theory. A disjunction may be true even though no disjunct is true.[14]

Since economists are typically unfamiliar with supervaluation theory, we will take the opportunity to present a more formal account of it. This account is taken from Fine (1975).

A More Formal Account

Let L denote some vague language. Fine suggests that a proper account of L requires the notion of a specification space S. This space contains a number of points. Each point in the specification space assigns a truth value to all of the propositions in L. Some points in the specification space are "partial" specification points. A partial specification point assigns a truth value - True, False, or Indefinite - to each proposition in L. By "Indefinite" we mean neither true nor false. To illustrate, in our previous example, "Jim is old" is assigned Indefinite at a particular partial specification point.

Points in the specification space are "admissible" if they assign truth values to the propositions of L that are in accordance with our intuitive understanding of the meaning of those propositions. To illustrate, points in the specification space that are admissible will always assign True to "Yul Brynner is bald", and False to "Mick Jagger is bald". However, admissible points in the specification space could also assign Indefinite to "Herbert is bald" where Herbert is a borderline case of baldness.[15] In the context of our earlier example, an admissible point would assign True to "Stonehenge is more impressive than the gospel chapel in Stoke Pewsey" and Indefinite to "Stonehenge is more impressive than Salisbury Cathedral". An admissible point can also be a partial specification point and vice versa.

Points in the specification space are partially ordered by a relation $\succeq$. A partial ordering is a reflexive, transitive and antisymmetric binary relation.[16]

For all $t, p \in S$ writing $t \succeq p$ means that t "extends" p. We say that t extends p if and only if t preserves truth whenever p does and also preserves falsity whenever p does. To illustrate, assume that p is a partial specification point. Let us also assume that p is admissible. Assume that p assigns True to "Yul Brynner is bald", and False to "Mick Jagger is bald". If t is an "extension" of p then it will preserve the truth values that are assigned to these two propositions by p. However, since t is an extension of p then it could also assign a definite truth value to the proposition "Herbert is bald". This proposition was indefinite at p but it could be assigned True at t. This is what is meant by an "extension".

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[14] This is an example of the failure of truth-functionality for compound propositions in supervaluation theory. The truth of a compound is not a function of the truth of its components. This is unlike classical logic and fuzzy logic. Alan Weir made the observation to us that if you are willing to reject bivalence then it is not entirely obvious why you should feel committed to the excluded middle law.

[15] These examples are taken from Fine 1975, 268.

[16] Reflexivity and transitivity were defined earlier. A binary relation $\succeq$ is antisymmetric if and only if, for all $x, y \in S$, $x \succeq y \wedge y \succeq x$ implies $x = y$.

Imagine that t assigns True to this proposition. Another extension of p however, u say, might assign False to this proposition. So we have $t \succeq p$ and $u \succeq p$, but neither t nor u extends the other. This is why the relation $\succeq$ is incomplete. It does not order all of the points in the specification space.

Fine says that a specification space can also be viewed as having a base point. This is a point from which all other specifications are extensions. Finally, Fine assumes that the specification space is "complete". A complete point in S is one that assigns only the values True and False to all of the propositions in L. At a complete point all vagueness has been resolved. A sharpening is simply a complete and admissible specification point.

We now consider another important characteristic of supervaluation theory. Fine calls this "penumbral connection".[17] Consider a specification point p. As before, assume that this point assigns True to "Yul Brynner is bald", False to "Mick Jagger is bald" and Indefinite to "Herbert is bald". One feature of these assignments is that they simultaneously determine the truth values that are assigned to other propositions like "Yul Brynner is not bald"; this must be false. Another example: if "Herbert is bald" is Indefinite then the same must be true for "Herbert is not bald". Therefore, assignments of truth values at any particular specification point constrain the truth values that can be assigned to other propositions in L at that point.[18]

This idea is taken up by Fine in his discussion of penumbral connection. He distinguishes between "external" and "internal" respectively.[19] First of all, here is an example of "external" penumbral connection. Consider a partial specification point p. Consider the proposition "This blob is red". Imagine that the blob in question is on the borderline between pink and red. Imagine too that at point p this proposition is assigned Indefinite. Consider now a sharpening of the language L that extends p. Imagine that in this sharpening the proposition "This blob is red" is assigned True. As we have just said, this means that the proposition "This blob is not red" must be assigned False. But, in addition, other propositions like "This blob is pink" must be false too. The reason for this is that if the blob in question is red in a sharpening, then it cannot at the same time be pink in *that* sharpening. Any sharpening of the predicate "red" will influence the boundary of the predicate "pink". Therefore, in any particular sharpening the propositions "This blob is red" and "This blob is pink" cannot both be true.

This means that the conjunction "This blob is red and this blob is pink" is false since it is false in all sharpenings. However the proposition "This blob is red" might well be neither true nor false, as might the proposition "This blob is pink". One implication of this which is commonly drawn by philosophers is that an adequate logic of vagueness cannot be truth-functional. This is particularly striking whenever we contrast supervaluation theory with the approach to vagueness taken by fuzzy logic (see the next section).

We now explain "internal" penumbral connection. Internal penumbral con-

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[17] Fine 1975, 270.

[18] Here is another example involving a comparative. If x is better than y at a point, then it is false that y is better than x at that point.

[19] Fine 1975, 275–276.

nection refers to how we treat different borderline cases of the same predicate in any particular sharpening. Imagine that Jim and John are both borderline cases of "old", but that Jim is older than John. Any sharpening that makes "John is old" true must also make "Jim is old" true. Of course a sharpening that makes "Jim is old" true does not necessarily make "John is old" true. One interesting consequence of this is the way supervaluation theory treats the sorites paradox. To recall, the first premise of the paradox is:

1. For all n, where n is a number, if "n grains of sand is not a heap" is true then "$n + 1$ grains of sand is not a heap" is also true.

Supervaluation theory rejects this premise. In any sharpening the conditional contained within the scope of the quantifier is false. Since the first premise is false in all sharpenings of the term "heap" then, according to supervaluation, it is plain false. This is how supervaluation solves the sorites paradox.

The importance of these connections can be highlighted when we consider how they apply to the case of comparatives. As we have already pointed out, if x is better than y at a point in S, then it is false that y is better than x at that point. Moreover, if at some point x is better than y and y is better than z, then x must be better than z at that point too.

How does supervaluation theory affect our analysis? In every sharpening of "Fer than" it will be the case that either a zone of indeterminacy exists that is hard, or no zone exists. In other words, each sharpening of "Fer than" is itself a comparative and the binary relations that represent these comparatives are either orderings or incomplete orderings. Crucially Broome demonstrates that at least one sharpening must be determinate, i.e. the binary relation that represents this sharpened comparative is complete.[20] If this were not the case, then we would have hard indeterminacy in every sharpening. Hard indeterminacy would then be true, according to supervaluation theory.

Welfare Economics

Applications of supervaluation theory in economics are rare. A notable exception is Broome's work on valuing changes in the population.[21] In addition to this work by Broome, it is remarkable how close at times Amartya Sen comes to adopting supervaluation theory in his writings.

For instance, in his important work on economic inequality Sen (1973) is interested in the structure of "more unequal than". One theme of this work is Sen's idea that the concept of inequality is inherently imprecise. He suggests that this imprecision in the concept of inequality means that we should not expect the binary relation that represents "more unequal than" to be complete. He recommends that the appropriate binary relation is the intersection of orderings, with each particular ordering corresponding to a particular measure of inequality.

[20] Broome 1997, 82.
[21] Broome 2004.

Sen calls the binary relation which emerges from this procedure an "intersection" quasi-ordering.[22]

It is remarkable how close Sen is to adopting supervaluation theory. We can interpret his argument that inequality is inherently imprecise as meaning that "more unequal than" is vague. Each particular measure of inequality that generates an ordering is a sharpening of this comparative. Sen claims that one distribution is more unequal than another if it is more unequal in all of the relevant measures of inequality. Supervaluation theory says that it is true that one distribution is more unequal than another if it is true in all sharpenings. So Sen and supervaluation theory agree in this case. In order for Sen to adopt supervaluation theory entirely, he would also need to make the following claim. He would need to say that it is true that one distribution is not more unequal than another if and only if that is true in all sharpenings. The comparative "more unequal than" which Sen seeks to represent would then have what Broome calls "soft", as opposed to "hard", indeterminacy. Only a small change to Sen's theory is needed to make it consistent with supervaluation theory.

3.2 Fuzzy Set Theory

Another approach to the logic of vague terms is taken by fuzzy set theory. Unlike supervaluation, this theory has found several applications in economics.[23]

A fuzzy set is the extension of a vague predicate. So if "poor" is vague, for example, then the set of poor people is a fuzzy set. Each object in the relevant domain is assigned a degree of membership to this fuzzy set. In the earliest version of the theory, these degrees are real numbers (weakly) between zero and one. If an object is assigned a degree of one then it definitely belongs to the set. If an object is assigned a degree of zero then it definitely does not belong to the set. However, there is a continuum of possible degrees of membership represented by the numbers between zero and one. In so-called "fuzzy logic" this number is also the degree of truth of the proposition "x is F", where x is the object in the domain and F is the vague predicate. If a proposition is assigned a degree of one then it is true and if it is assigned a degree of zero then it is false.

Rather than using $[0, 1]$ ordered by $\geq$, it is possible to assign degrees of membership that belong to a (possibly finite) set whose elements are ranked by an ordering T. These degrees could correspond to linguistic phrases such as "not at all", "a little", "possibly", "very much" and "definitely". The idea of replacing $[0, 1]$ with an ordered set is due to Goguen (1967) and is called an "ordinally fuzzy

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[22] Sen 1973, ch 3. Note that in Sen's procedure we take the intersection of orderings, whereas for Rabinowicz's parity modelling each quasi-ordering in his class may be incomplete.

[23] For instance there is a large literature on modelling fuzzy preferences, and also on individual and social choice when preferences are fuzzy. This literature is surveyed in Salles 1998 and Barrett and Salles (forthcoming) and so it is unnecessary to provide another survey of these applications here. For more details on specific topics see Barrett and Pattanaik 1985; 1989; Barrett et. al. 1986; 1992; Basu et. al. 1992, Dasgupta and Deb 1996 and Perote-Peña and Piggins 2007.

framework" by Barrett et al. (1992). It is sometimes referred to as a "linguistic fuzzy framework".

Can fuzzy set theory help with our analysis of soft indeterminacy? As we move from the better zone into the zone of indeterminacy, the proposition "This point is better than the standard" switches from true to not true, but it is not false either. The degree of truth of this proposition will be less than one but greater than zero. Similarly, the truth value of the proposition "The standard is better than this point" will also be greater than zero but less than one. A problem now arises with a more general version of Broome's collapsing principle. This principle says that for any x and y, if it is more true that x is Fer than y than that y is Fer than x, then x is Fer than y.[24] Since degrees of truth lie in $[0, 1]$ and are ordered by $\geq$, if we accept this more general version of the collapsing principle then the zone of indeterminacy cannot exist. As we mentioned earlier, the collapsing principle is controversial. If we accept it then degrees of truth cannot be numerical and ordered by $\geq$, instead we need a version of Goguen's theory in which the set of degrees is incompletely ordered by a binary relation T^*. Such a theory could then provide a representation of soft indeterminacy that meets Broome's constraint.

Philosophical Objections

Philosophers often claim that there are problems with fuzzy logic.[25] The most commonly stated problem is the failure to respect penumbral connections. Fuzzy set theory has a rule concerning how the degree of membership of an object to two fuzzy sets is determined by the degree of membership of the object in each of these fuzzy sets. The degree to which an object belongs to the intersection of two fuzzy sets is determined by the smaller of the two individual degrees of membership. Loosely speaking, if an object is red to a degree and round to a degree, then the degree to which it is both red and round is the smaller of these two individual degrees. Similarly, in fuzzy logic, the degree of truth of a conjunction is determined by the minimum degree of truth of the conjuncts. As we mentioned earlier, this is often called the view that vagueness is "truth-functional".

However, many philosophers claim that when dealing with vague predicates truth-functionality is mistaken. Here is an example. Imagine that a person is borderline poor. On the fuzzy account, this person is poor to a degree. Similarly, the person is not poor to a degree. Consider the conjunction "This person is poor and not poor". This must be false since it is a logical contradiction. Indeed, if we accept supervaluation theory, it is false because it is false in all sharpenings. However, fuzzy logic suggests that the degree of truth of a conjunction is simply the smaller of the degrees of truth of the conjuncts. So this proposition might not be false according to fuzzy logic.

What about disjunctions? The degree to which an object belongs to the

[24] Broome 1997, 77.

[25] Strong objections are presented in Williamson 1994; 2003.

union of two fuzzy sets is determined by the larger of the two individual degrees of membership. Again, if an object is red to a degree and round to a degree, then the degree to which it is red or round is the larger of these two individual degrees. Similarly, in fuzzy logic, the degree of truth of a disjunction is the larger degree of truth of the disjuncts. So, unlike supervaluation, the disjunction "This person is poor or not poor" may be less than true. Excluded middle does not hold in fuzzy logic.

These are standard objections. However, as we mentioned earlier in footnote 14, if you are willing to abandon bivalence then it is not immediately obvious why you should accept excluded middle and non-contradiction.

4. Conclusion

In cases where our ability to compare a pair of things breaks down, we commonly say that it is false that either is better than the other and false that they are equally good. Following Broome, we have called this "hard indeterminacy". Our purpose in this paper has been to survey alternative representations. We have considered the possibility of parity, which has proven to be popular in philosophical circles, and described some ways of formally expressing this idea. We have also considered the idea that our failure to compare things is just a consequence of vagueness. We contrasted two theories of vagueness; supervaluation theory and fuzzy set theory.

What are the consequences of this work for economics? One is particularly striking. If the ideas surveyed in this paper are correct then preference indeterminacy (both individual and social) is misrepresented; it does not take the form that is commonly assumed.

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