Analyse & Kritik 29/2007 (© Lucius & Lucius, Stuttgart) p. 290–310

Prasanta K. Pattanaik/Yongsheng Xu

On Measuring Personal Connections and the Extent of Social Networks*

Abstract: The notions of personal connection and social networks are key ingredients of the increasingly important concept of social capital in social sciences in general and in economics in particular. This paper discusses the problem of measuring personal connection and the extent of social networks that may exist in a society. For this purpose we develop several conceptual and analytical frameworks. In the process, we axiomatically characterize several measures of personal connection and social networks.

0. Introduction

The purpose of this paper is to develop measures of the extent of social networks that may exist in a society. In the process, we also develop measures of the degree of personal connection between two individuals, the notion of personal connection between individuals being the conceptual basis of our notion of social networks.

The concept of social networks is one of three key concepts, trust, norms and networks, which have been the focus of a number of important contributions by sociologists (see, among others, Putnam (1993; 1995) and Coleman (1990; 2000) and economists (see, for example, Stiglitz 2000 and Dasgupta 2000) in the literature on 'social capital'. While it is not entirely clear that 'social capital' is the most appropriate term for describing collectively these three elements, there seems to be general agreement that they have important consequences for the functioning of a society. For example, social networks may make a person's life richer and happier (thus serving as a 'consumption good'). They can facilitate transactions and cooperative ventures by building trust, and can serve as conduits for the flow of information. They can also serve as a type of informal insurance insofar as one may fall back on one's personal connections in the case of certain emergencies. Of course, social networks need not always play a benign role: social networks can be used to oppress those outside the network and to promote factionalism.

Despite the increasing attention that the concept of social networks as a component of social capital has recently received, there does not seem to be much

* We are indebted to Thomas Kelly and Sunder Ramaswamy for introducing us to the literature on social capital. For helpful comments we are grateful to Marlies Ahlert, Kaushik Basu, Rajat Deb, Wulf Gaertner, Michael Jones, Marc Kilgour, and Joyashree Roy.

formal treatment of the issue of measuring the extent of social networks.[1] In this paper, we develop an approach to the problem of measuring social networks. In doing so, we also derive measures of personal connection between individuals. Since our focus is the problem of measuring personal connection between individuals and the extent of social networks that already exist in a society, we take these connection and networks as given and do not discuss how they come into existence (for a discussion of how social networks evolve, see, among others, Dutta and Jackson 2001). It may be worth noting that, our measures of personal connection and social networks are 'descriptive' in character: they do not involve any judgement about whether personal connection and social networks promote individual and/or social welfare.

Our analysis proceeds in two stages. First, we discuss the problem of measuring personal connection between two individuals. In our framework, this is necessary for measuring the extent of social networks. As Scott (1991, 3) observes, the analysis of social network must be based on relational data, "the contacts, ties and connections, the group attachments which relate one individual to another, and so cannot be reduced to the properties of the individual agents themselves". Intuitively, our notion of personal connection between two individuals reflects the friendly relation that may exist between the two individuals directly or indirectly: the two individuals may have a direct friendly relation or they may have friends who know of each other through their friends. Either way, one can think of a benign chain of friends linking these two individuals. From this perspective, the measurement of the degree of personal connection between two individuals involves an examination of the set of all benign chains between them. Using this intuition, we first axiomatically characterize two measures of the degree of personal connection between two individuals based on the length of the shortest benign chains between them (as we note in Remark 1.2 below, our notion of a shortest benign chain is closely related to the notion of a *geodesic* in the literature on social networks, though the two concepts are not identical). These measures have an interesting feature: the degree of personal connection between two individuals is (weakly) inversely related to the length of the shortest benign chains. We then characterize several other measures of personal connection, using the specific interpretation of personal connection as a means of transmitting messages. These other measures capture the idea that benign chains are conduits of messages from one person to another and that, when a message is transmitted indirectly from one person to another via a benign chain, it gets diluted at each successive stage in the transmission process.

In the second stage of our analysis, we develop measures of social networks, using personal connections between individuals as building blocks. We view the problem of measuring social networks as a problem of aggregation, namely the aggregation of personal connections for all pairs of distinct individuals in the

[1] There have been a number contributions investigating conceptually and empirically the relevance of social capital to economics and the measurement of trust and trustworthiness–two key components of social capital. See, for example, Bowles and Gintis 2002; Durlauf 2002; Glaeser, Laibson and Sacerdote 2002; Glaeser, Laibson, Scheinkman and Soutter 2000; Knack and Keefer 1997; LaPorta, Lopez-de-Salanes, Shleifer and Vishny 1997.

society. This procedure fits in well with some suggestions for the measurement of social networks that one finds in the literature. Indeed, as Marsden (1990) indicates, network range, the strength of personal connections, network size and density are some of the indicators of the extent of social networks. Many of these indicators are incorporated in our procedure of measuring the degree of personal connections for each pair of distinct individuals in the society and then aggregating these personal connections to arrive at a measure of social networks. We show that, under certain plausible conditions, the extent of social networks existing in a society is the simple sum of the degrees of personal connections for all pairs of distinct individuals in the society.

The plan of our paper is as follows. In Section 1, we present the basic notation and definitions. In Section 2, we axiomatically characterize some measures of the degree of personal connections between two individuals. Section 3 proposes and axiomatically characterizes some measures of the extent of social networks in a given society. We conclude in Section 4. Proofs of our results are organized in Appendices A, B, and C.

1. Basic Notation and Definitions

Let $N = \{1, 2, \cdots, n\}$ be a community of $n \geq 3$ individuals. Let R and $R(-)$ be two binary relations defined over N, such that: (i) R is reflexive and symmetric; (ii) $R(-)$ is irreflexive and symmetric; and (iii) for all i and j in N, iRj implies not$[iR(-)j]$.[2] For reasons that will be obvious from the interpretations attached to R and $R(-)$, R and $R(-)$ are not assumed to be either necessarily connected or necessarily transitive. The interpretation of R is as follows. For all i and j in N, iRj denotes "i has a good relation with j"[3]. Similarly, $iR(-)j$ denotes that i has a hostile relation with j.[4]

Note that, under our specification, not(iRj) does not necessarily imply $iR(-)j$: two individuals i and j may have neither a good relation nor a hostile relation between them. When iRj, i and j have a direct (benign) personal connection.

---

[2] The two binary relations, R and $R(-)$, constitute the basic building blocks of our formal model. It has been suggested to us that some of the things that we do in terms of R and $R(-)$ could be couched in the language of graph theory. However, we have not followed this suggestion for two reasons. First, we found that, by using the graph-theoretic language, we would not gain much in terms of economy in our exposition. Secondly, we felt that, compared to an exposition in terms of graph theory, our present exposition may have the advantage of being accessible to a wider group of social scientists.

[3] Some of our colleagues have occasionally raised questions about the symmetry of the relation "has a good relation with". It has been pointed out to us that a 'similar' binary relation, "is in love with" is not necessarily symmetric. While symmetry is certainly an assumption of doubtful validity in the case of the binary relation "is in love with", the situation seems to be very different for the binary relation "has a good relation with". The assumption that, if i has a good relation with j, then j must have a good relation with i, seems very plausible to us. Similarly, we find it compelling to assume that the binary relation "has a hostile relation with" is symmetric.

[4] The data relating to our two binary relations, R and $R(-)$, can be collected by the Moreno 1934 procedure, familiar in social network analysis. Moreno 1934 used a form of his sociometric experiment to record not only friendship but enmity as well.

However, even if i may not have any direct personal connection with j, i may have a good relation with someone who, in turn, may have a good relation with j. Indeed, in general, one can think of a "good relation chain" linking i and j through a series of individuals functioning as intermediaries between i and j.[5] In the absence of a direct personal connection between i and j, "such a good relation chain" or indirect personal connection can, to some extent, serve some of the desirable or undesirable purposes that a direct (benign) personal connection does. An indirect personal connection can enrich a person's life, though to a lesser extent than a direct personal connection. An indirect personal connection can also promote trust and facilitate economic transactions. It can also act as a conduit for the flow of information, though it would, presumably be a less effective conduit than a direct personal connection. Finally, in case of necessity, one can also appeal to another individual for help, using an indirect personal connection. These considerations provide the motivation for our next definition.

Definition 1.1: For all distinct i and j, we say that there exists a *benign chain* from i to j iff there exists a positive integer t ($t \leq n$) such that, for some $m(1), \cdots, m(t)$ in N, we have:

(1.1.1) $m(1) = i, m(t) = j$;

(1.1.2) $m(1), ..., m(t)$ are all distinct;

(1.1.3) for every positive integer $k < t, m(k)Rm(k+1)$;

(1.1.4) for all i' and j' in $\{m(1), ..., m(t)\}$, not$[i'R(-)j']$.

Given such $m(1), \cdots, m(t)$, we call the finite sequence $(m(1), \cdots, m(t))$ a *benign chain* from i to j, and we define the *length of the chain* to be $(t-1)$ (i.e., the number of "elementary links" in the chain). A *shortest benign chain* from i to j is a benign chain from i to j that has the smallest length.

Remark 1.2: The notion of a shortest benign chain is similar to, though not identical with, the notion of a *geodesic* in the literature on social networks. The main difference between the two concepts lies in the fact that our definition of a benign chain requires that no two individuals involved in the 'chain' should have a hostile relation, while this requirement is not incorporated in the notion of a *path*, on which the notion of a geodesic is based. It may be worth noting that the geodesic is an important concept in the literature on social networks: it is often taken as a measure of how far apart two individuals are; it has been used in several of the centrality measures in social network analysis; it is an important factor for constructing some particular social networks; and, in some writings on communication networks, it has been *assumed* that the message betweeen two individuals is transmitted through a geodesic between them. (See Wasserman and Faust 1994 for a discussions of the importance of geodesics in social network analysis.)

[5] In our more formal definition (Definition 1.1) of a "benign chain", we require that no two individuals involved in the chain should have a hostile relation.

Remark 1.3: It can be checked that: (i) if $(m(1), \cdots, m(t))$ is a benign chain from i to j, then $(m(t), \cdots, m(1))$ must be a benign chain from j to i; (ii) the length of a benign chain from i to j cannot exceed $n-1$; and (iii) the smallest length that a benign chain from i to j can possibly have is 1 (this happens when iRj).

When there is a benign chain from i to j, we say that i and j are *linked*; otherwise, we say that i is *isolated* from j.

2. Personal Connections

In this section, we discuss some measures of the closeness of the favourable direct or indirect relation that may exist between two distinct individuals i and j. Let Z be the class of all two-element subsets of N.

Let $\wp$ be the collection of *all* $(R, R(-))$ such that R and $R(-)$ are two binary relations defined over N with the properties specified in Section 2. The elements in $\wp$ will be denoted as $\mathbb{R}, \mathbb{R}'$, etc. An $\mathbb{R} \in \wp$ is said to be *without hostility* iff there exists no $\{i,j\} \in Z$ such that $iR(-)j$.

Let $\mathbb{R} \in \wp$ be without hostility. If, under $\mathbb{R}$, $(m(1) = i, \cdots, m(t) = j)$ is a shortest benign chain from i to j, then for all positive integers i' and j' with $1 \le i' < j' \le t$, $(m(i'), \cdots, m(j'))$ is a shortest benign chain from $m(i')$ to $m(j')$.

Let $d : \wp \times Z \to (-\infty, +\infty)$ be a function from $\wp \times Z$ to the real line. The intended interpretation of the d function can be explained as follows: For a given $\mathbb{R}$ and any two distinct individuals i and j in N, $d(\mathbb{R}, \{i,j\})$ denotes the degree of personal connection between i and j, or the extent of the favourable (direct or indirect) relation that may exist between i and j, given $\mathbb{R}$. Therefore, for a given $\mathbb{R}$, for all $\{i,j\}$ and $\{p,q\}$ in Z, $d(\mathbb{R}, \{i,j\}) \ge d(\mathbb{R}, \{p,q\})$ will be interpreted as meaning that, given $\mathbb{R}$, the degree of the personal connection between i and j is at least as great as the degree of the personal connection between p and q.

2.1 Some General Properties of Personal Connections

In this subsection, we discuss some general properties of the $d(\cdot,\cdot)$ function. For this purpose, we consider the following properties imposed on $d(\cdot,\cdot)$.

Definition 2.1 $d(\cdot,\cdot)$ satisfies:

(2.1.1) *Simple Domination (I)* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i,j\}, \{p,q\} \in Z$, if $[iRj$ or p is isolated from q under $\mathbb{R}']$ then $d(\mathbb{R}, \{i,j\}) \ge d(\mathbb{R}', \{p,q\})$;

(2.1.2) *Simple Domination (II)* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i,j\}, \{p,q\} \in Z$, if $[iRj$ and not$(pR'q)]$ or $[i$ is linked with j under $\mathbb{R}$, and p is isolated from q under $\mathbb{R}']$ then $d(\mathbb{R}, \{i,j\}) > d(\mathbb{R}', \{p,q\})$;

(2.1.3) *Weak Simple Domination (II)* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i,j\}, \{p,q\} \in Z$, if [iRj and not$(pR'q)$] or [i is linked with j under $\mathbb{R}$ and p is isolated from q under $\mathbb{R}'$] then $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{p,q\})$;

(2.1.4) *Independence* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i,j\} \in Z$ be such that $\mathbb{R}$ and $\mathbb{R}'$ are without hostility, if $(m(1) = i, \cdots, m(s), m(s+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}$, $(m'(1) = i, \cdots, m'(t), m'(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}'$, then,

$$d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\}) \Leftrightarrow d(\mathbb{R}, \{i, m(s)\}) \geq d(\mathbb{R}', \{i, m'(t)\});$$

(2.1.5) *Weak Independence* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i,j\} \in Z$ be such that $\mathbb{R}$ and $\mathbb{R}'$ are without hostility, if $(m(1) = i, \cdots, m(s), m(s+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}$, $(m'(1) = i, \cdots, m'(t), m'(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}'$, then,

$$d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\}) \Rightarrow d(\mathbb{R}_1, \{i, m(s)\}) \geq d(\mathbb{R}_1', \{i, m'(t)\});$$

(2.1.6) *Neutrality* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$ and for all $\{i,j\} \in Z$, if the set of all benign chains from i to j under $\mathbb{R}$ is the same as the set of all benign chains from i to j under $\mathbb{R}'$, then $d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}', \{i,j\})$;

(2.1.7) *Anonymity* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, and all one-to-one function σ from N to N, if, for all $i, j \in N$, [(iRj) iff $\sigma(i)R'\sigma(j)$] and [$iR(-)j$ iff $\sigma(i)R'(-)\sigma(j)$], then, for all $\{i,j\} \in Z, d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}', \{\sigma(i), \sigma(j)\})$;

(2.1.8) *Monotonicity* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$ and all $\{i,j\} \in Z$, if every benign chain from i to j under $\mathbb{R}'$ is a benign chain from i to j under $\mathbb{R}$, then $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\})$;

(2.1.9) *Dominance* iff, for all $\mathbb{R}, \mathbb{R}', \mathbb{R}'' \in \wp$ and all $\{i,j\} \in Z$, if [the set of all benign chains from i to j under $\mathbb{R}''$ is the union of the set of all benign chains from i to j under $\mathbb{R}$ and the set of all benign chains from i to j under $\mathbb{R}'$] and [$d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\})$], then $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}'', \{i,j\})$.

Simple Domination (I) requires that, if i and j are directly connected under $\mathbb{R}$ or p and q are isolated from each other under $\mathbb{R}'$, then the degree of personal connection between i and j under $\mathbb{R}$ is at least as great as the degree of personal connection between p and q under $\mathbb{R}'$. It is a plausible axiom, but it may be worth noting that Simple Domination (I) rules out certain types of intuition. Suppose iRj but (i,j) is the only benign chain between i and j. On the other hand, suppose not[$i'Rj'$] but there exist 100 distinct individuals, $p_1, ..., p_{100}$, such that, for $t = 1, ..., 100, (i', p_t, j')$ constitutes a benign chain from i' to j'. Then, for some intuitive purposes, the connection between i' and j' may be considered closer than the connection between i and j. For example, if i' wants to induce j' to do something for him, then i' can get 100 different individuals to intercede with j' for him, and that may be even more effective than the persuasive influence that i can exert on j through his direct friendly relation with j. Thus, intuitively, one may like to admit the possibility that $d(\mathbb{R}, \{i', j'\}) > d(\mathbb{R}, \{i,j\})$

in this case. However, this is not permissible under Simple Domination (I). Simple Domination (II) extends the intuition embedded in Simple Domination (I) further by requiring that the degree of personal connection between i and j under $\mathbb{R}$ is greater than the degree of personal connection between p and q under $\mathbb{R}'$ if either i is directly connected with j under $\mathbb{R}$ and p is not directly connected with q under $\mathbb{R}'$, or i is linked with j under $\mathbb{R}$ while p is isolated from q under $\mathbb{R}'$. In our current framework, Simple Domination (II) seems plausible as well, but the reservation that we noted in the case of Simple Domination (I) also applies to Simple Domination (II). Weak Simple Domination (II) is a weaker requirement than Simple Domination (II). It should also be noted that Simple Domination (I) implies Weak Simple Domination (II).

Independence requires the following. Suppose $\mathbb{R}$ and $\mathbb{R}'$ are without hostility. Suppose there is a unique benign chain from i to j under each of $\mathbb{R}$ and $\mathbb{R}'$. Let k be the individual immediately before j in the benign chain from i to j under $\mathbb{R}$, and let k' be the individual immediately before j in the benign chain from i to j under $\mathbb{R}'$. Then Independence requires that the ranking of the degree of personal connection between i and k under $\mathbb{R}$ and the degree of personal connection between i and k' under $\mathbb{R}'$ must be analogous to the ranking of the degree of personal connection between i and j under $\mathbb{R}$ and the ranking of the personal connection between i and j under $\mathbb{R}'$. Weak Independence is a weaker version of independence.

Essentially, Neutrality stipulates that the degree of personal connection between two individuals depends only on the set of benign chains between those two individuals. If, in switching from $\mathbb{R}$ to $\mathbb{R}'$, the set of benign chains from i to j remains the same, then the degree of personal connection between i and i' will remain unchanged. Anonymity rules out the possibility that some people may be more "effective" in a benign chain as compared to other people.

Monotonicity reflects the intuition that the degree of personal connection between any two individuals does not decrease when additional benign chains between them come into existence. Dominance has a very different type of underlying intuition. Let $\mathbb{R}$ and $\mathbb{R}'$ be such that the degree of personal connection between i and j under $\mathbb{R}$ is at least as great as the degree of personal connection between i and j under $\mathbb{R}'$, and let $\mathbb{R}''$ be such that the set of benign chains between i and j under $\mathbb{R}''$ is simply the union of the two sets of benign chains between i and j under $\mathbb{R}$ and $\mathbb{R}'$. Now compare the degrees of personal connection between i and j under $\mathbb{R}$ and $\mathbb{R}''$. Note that, in going from $\mathbb{R}$ to $\mathbb{R}''$, we are merging with the set of already existing benign chains between i and j another set of benign chains (namely, those that exist under $\mathbb{R}'$), which is not 'superior' to or 'more effective' than the set of already existing benign chains (we know this since the degree of personal connection between i and j is no greater under $\mathbb{R}'$ than under $\mathbb{R}$). Given this, Dominance requires that the degree of personal connection between i and j should not increase when we make the transition from $\mathbb{R}$ to $\mathbb{R}''$. How sound is this intuition? Suppose the sole purpose for which a benign chain may be used is to convey messages from the person at the beginning of the chain to the person at the end of the chain, and that in sending a message to another person, the originator of the message chooses only one chain. Then, it is reasonable to assume that the value of a set of benign chains is simply the value of those individual benign chains in the set, which are most effective for this purpose. In that case, it is reasonable to postulate that, given our specifications of $\mathbb{R}$, $\mathbb{R}'$, and $\mathbb{R}''$, there is no gain in terms of the degree of personal connection between i and j when we switch from $\mathbb{R}$ to $\mathbb{R}''$. However, one can think of alternative scenarios where one's intuitiuon may go in a different direction. Suppose, as in the counterexample that we considered in the case of Simple Domination (I), benign chains are used to exert influence on other individuals. In that case, merging the set of benign chains between i and j under $\mathbb{R}'$ with the set of benign chains between them under $\mathbb{R}$ will provide i with an opportunity to exert more persuasive influence on j, even though, considered separately, the former set of benign chains was no more effective than the latter set of benign chains. In that case, Dominance can be violated. Thus, as in the case of Simple Domination (I), and Simple Domination (II), the appeal of Dominance depends on how we visualize the purpose of benign chains. In this paper, our main emphsis is on the use of benign chains simply to transmit messages. In this specific context, Dominance, as well as Simple Domination (I) and Simple Domination (II), seems to have considerable appeal.

With the help of the above properties imposed on d, we are ready to present the following results. Their proofs can be found in Appendix A.

Theorem 2.2. d satisfies Simple Domination (I), Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$ and all $\{i, j\}, \{p, q\} \in Z$,

$$d(\mathbb{R}, \{i, j\}) \geq d(\mathbb{R}', \{p, q\}) \text{ iff } ([t \leq s] \text{ or } [p \text{ is isolated from } q \text{ under } \mathbb{R}']) \quad (1)$$

where t is the length of a smallest benign chain from i to j under $\mathbb{R}$ and s is the length of a smallest benign chain from p to q under $\mathbb{R}'$.

Theorem 2.3. d satisfies Simple Domination (I), Neutrality, Anonymity, Weak Independence, Monotonicity and Dominance if and only if for all $\mathbb{R}, \mathbb{R}' \in \wp$, all $\{i, j\}, \{p, q\} \in Z$,

$$[t \leq s] \text{ or } [p \text{ is isolated from } q \text{ under } \mathbb{R}'] \Rightarrow d(\mathbb{R}, \{i, j\}) \geq d(\mathbb{R}', \{p, q\})$$

where t is the length of the shortest benign chain from i to j under $\mathbb{R}$, and s is the length of the shortest benign chain from p to q under $\mathbb{R}'$.

2.2 Personal Connections Interpreted in Terms of Message Transmission

In the preceding section, we developed a general measure of the degree of personal connection: we showed that certain axioms characterize the ranking of degrees of personal connection on the basis of the length of the shortest benign chains involved. In the rest of Section 2, we develop some other measures of personal connection, using the specific interpretation of personal connection as a means of transmitting messages. Under this interpretation, benign chains will be viewed as conduits of messages from one person to another. A central idea in our context is that when a message is transmitted from i to j via a benign chain, if the message has to go through other individual(s) to reach j, the message will be diluted in the process of transmission.

Throughout this subsection, we make the following assumption:

Normalization. For all $\mathbb{R} \in \wp$ and all $\{i, j\} \in Z$, if iRj then $d(\mathbb{R}, \{i, j\}) = 1$ and if i is isolated from j under $\mathbb{R}$, then $d(\mathbb{R}, \{i, j\}) = 0$.

Let $\mathbb{R} = (R, R(-)) \in \wp$ be given. Let $\{i, j\} \in Z$ and let $C = (m(1) = i, \cdots, m(t+1) = j)$ be a benign chain from i to j with $t \geq 1$. We now introduce the notion of C being used as a *message channel* from i to j. Suppose $i = m(1)$ sends a message of volume 1 to $j = m(t+1)$, using C as a message channel. Then: (i) for every $k \in \{2, \cdots, t+1\}$, $m(k)$ receives a message in the amount of $a_{m(k)}[C]$ ($a_{m(k)}[C] \geq 0$) from $m(k-1)$, the predecessor of $m(k)$ in the benign chain (the amount of the message received by $m(k)$ is the same as the amount of the message sent by $m(k-1)$ to $m(k)$); (ii) for every $k \in \{2, \cdots, t\}$, an amount $a^L_{m(k)}[C]$ ($a^L_{m(k)}[C] \geq 0$) of the message received by $m(k)$ is lost; and (iii) for every $k \in \{3, \cdots, t+1\}$, $a_{m(k)}[C] = a_{m(k-1)}[C] - a^L_{m(k-1)}[C]$.

Let $\mathbb{R} \in \wp$ be given. The next property concerns the efficiency of message channels. Given our interpretation of benign chains as potential message channels, it seems natural to assume that the maximum amount of a unit message, which can be transmitted from i to j, constitutes a measure of the personal connection between i and j. This is the intuition underlying the following assumption.

Assumption 2.4. For all $\mathbb{R} \in \wp$ and all $\{i, j\}$ in Z, if, according to $\mathbb{R}$, there is a benign chain from i to j, then, for some benign chain $C = (m(1) = i, \cdots, m(t+1) = j)$ with $t \geq 1$ from i to j, $d(\mathbb{R}, \{i, j\})$ is the amount of message received by j from her predecessor $m(t)$, when i uses C to send a message of volume 1 to j.

When transmitting messages indirectly from i to j, our message channel may contain noise: some portion of the message may get lost at each stage of transmission. The lost amount of the message may be regarded as reflecting *frictions* for a given society. We distinguish two scenarios here. In the first scenario, the amount of the message that gets lost at each stage of the transmission process is exogenously given. In the second scenario, the amount of the message that is lost at each stage is endogenously determined.

Exogenously Determined Message Loss

There are two plausible ways of looking at the indirect transmission of a message from i to j when the loss of message at each stage of indirect transmission is exogenously determined. In the first instance, one may assume that, when a message is indirectly transmitted through a benign chain, at each stage of this indirect transmission, an exogenously given *fraction* of the received message is lost and the rest is transmitted to the next stage. Alternatively, one may assume that, when a message is indirectly transmitted through a benign chain, at each stage of this indirect transmission, an exogenously given *absolute amount* is lost and the rest is transmitted to the next stage. We shall consider both ways and derive some implications. The proofs of our results in this subsection can be found in Appendix B.

Exogenously Determined Proportional Message Loss. We say that the process of message transmission is characterized by *exogenously determined proportional message loss* iff there exists $\psi \in (0,1)$ such that, for every $\mathbb{R} \in \wp$, all $\{i,j\} \in Z$, and every benign chain $C = (m(1) = i, m(2), \cdots, m(t+1) = j)$ from i to j under $\mathbb{R}$, if i sends a message of volume 1 to j through the message channel C, then

$$a^L_{m(k)}[C] = \psi a_{m(k-1)}[C], \text{ for all } k \in \{3, \cdots, t+1\}.$$

Exogenously Determined Absolute Message Loss. We say that the process of message transmission is characterized by *exogenously determined absolute message loss* iff there exists $\alpha > 0$, such that, for every $\mathbb{R} \in \wp$, all $\{i,j\} \in Z$, and every benign chain $C = (m(1) = i, m(2), \cdots, m(t+1) = j)$ from i to j under $\mathbb{R}$, if i sends a message of volume 1 to j through the message channel C, then

$$a^L_{m(k)}[C] = \max\{\min\{a_{m(k-1)}[C] - \alpha, \alpha\}, 0\}, \text{ for all } k \in \{3, \cdots, t+1\}.$$

Theorem 2.5. Suppose Normalization and Assumption 2.4 are satisfied and the process of message transmission is characterized by exogenously determined proportional message loss, where the fraction of message lost at each stage is given by ψ. Then $d(\cdot,\cdot)$ satisfies Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance iff, for all $\mathbb{R} \in \wp$ and all $\{i,j\}$ in Z, we have:

(i) $d(\mathbb{R}, \{i,j\}) = 0$, if i and j are isolated under $\mathbb{R}$;

(ii) $d(\mathbb{R}, \{i,j\}) = 1$ if iRj, and

(iii) $d(\mathbb{R}, \{i,j\}) = (1-\psi)^t$, where $t \geq 1$ and $t+1$ is the length of a shortest benign chain from i to j under $\mathbb{R}$.

Theorem 2.6. Suppose Normalization and Assumption 2.4 are satisfied and the process of message transmission is characterized by exogenously determined absolute message loss, where the absolute amount of message lost at each stage is given by $\alpha > 0$. Then, $d(\cdot,\cdot)$ satisfies Weak Simple Domination (II), Weak Independence, Neutrality, Anonymity, Monotocity and Dominance iff, for all $\mathbb{R} \in \wp$ and all $\{i,j\}$ in Z, we have:

(i) $d(\mathbb{R}, \{i,j\}) = 0$, if i and j are isoloated under $\mathbb{R}$;

(ii) $d(\mathbb{R}, \{i, j\}) = 1$ if iRj, and

(iii) $d(\mathbb{R}, \{i, j\}) = \max\{1 - t\alpha, 0\}$, where $t \geq 1$ and $t + 1$ is the length of a shortest benign chain from i to j under $\mathbb{R}$.

Endogenously Determined Message Loss

In this subsection, we consider the possibility that the amount of message loss at each stage of indirect transimission is determined endogenously.

Endogenously Determined Message Loss. We say that the process of message transmission is characterized by *endogenously determined message loss* iff, for every $\mathbb{R} \in \wp$, all $\{i, j\} \in Z$, and every benign chain $C = (m(1) = i, m(2), \cdots, m(t+1) = j)$ from i to j under $\mathbb{R}$, when i sends a message of volume 1 to j through C, we have

$$a^L_{m(k)}[C] = a^L_{m(k+1)}[C] = a_{m(k)} \text{ for every } k \in \{2, \cdots, t\},$$

that is, the same amount of the message is lost at each stage of indirect transmission and j receives the same amount of the message as is lost by $m(t)$.

The following theorem summarizes the implication of the framework with endogenously determined message loss. The proof of the theorem can be found in Appendix B.

Theorem 2.7. Suppose Normalization and Assumption 2.4 are satisfied and the process of message transmission is characterized by endogenously determined message loss. Then, $d(\cdot, \cdot)$ satisfies Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance iff, for all $\mathbb{R} \in \wp$ and all $\{i, j\}$ in Z,

(i) $d(\mathbb{R}, \{i, j\}) = 0$ if i and j are isolated;

(ii) $d(\mathbb{R}, \{i, j\}) = 1$ if iRj; and

(iii) $d(\mathbb{R}, \{i, j\}) = 1/t$ where $t \geq 1$ and t is the length of a smallest benign chain from i to j under $\mathbb{R}$.

3. The Extent of Social Networks

In this section, we discuss the issue of measuring the extent of social networks for different societies. For a given $\mathbb{R} \in \wp$, intuitively, the extent of networks in $\mathbb{R}$ can be thought of as a function of the degrees of benign connections for all two distinct $\{i, j\}$ in Z. We use $\omega(\mathbb{R})$ to denote the extent of social networks in the society given by $\mathbb{R}$. Consider the following axioms to be imposed on $\omega(\cdot)$.

Definition 3.1. ω is said to satisfy

(3.1.1) *Marginal Contribution* iff, for all $\mathbb{R}, \mathbb{R}' \in \wp$, for all $\{i,j\} \in Z$, if for all $\{p,q\} \in Z$,

$$\begin{aligned} & [d(\mathbb{R}, \{p,q\}) \geq d(\mathbb{R}, \{i,j\}),] \\ & [\{p,q\} \neq \{i,j\} \Rightarrow ((pRq \Leftrightarrow pR'q) \text{ and } (pR(-)q \Leftrightarrow pR'(-)q))], \\ \text{and} \quad & [iR'(-)j\text{and not}(iR(-)j)], \end{aligned}$$

then

$$\omega(\mathbb{R}) - \omega(\mathbb{R}') = d(\mathbb{R}, \{i,j\});$$

(3.1.2) *ω-Normalization* iff, for all $\mathbb{R} \in \wp$, if for all $\{i,j\} \in Z$, not(iRj), then $\omega(\mathbb{R}) = 0$.

ω-Normalization is simply a convention and does not impose any significant restriction on the extent of social networks. Marginal Contribution stipulates that, if the only difference between $\mathbb{R}$ and $\mathbb{R}'$ is that two individuals, i and j, are non-hostile and have the "weakest" link under $\mathbb{R}$ and are hostile and have no link under $\mathbb{R}'$, then the change in the extent of social networks when we switch from $\mathbb{R}'$ to $\mathbb{R}$ is captured by the degree of the personal connection between these two individuals under $\mathbb{R}$: after all, there is absolutely no benign chain from i to j under $\mathbb{R}'$ and hence the degree of the personal connection between them under $\mathbb{R}'$ is zero.

We now state our results in this section. Their proofs can be found in Appendix C.

Theorem 3.2. Suppose d satisfies Simple Domination (I), Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance, and ω satisfies Marginal Contritution and ω-Normalization. Then,

$$\text{for all } \mathbb{R} \in \wp, \omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i,j\}),$$

where $d(\cdot,\cdot)$ has the property given in Theorem 2.2.

Theorem 3.3. Suppose d satisfies Simple Domination (I), Neutrality, Anonymity, Weak Independence, Monotonicity and Dominance, and ω satisfies Marginal Contritution and ω-Normalization. Then,

$$\text{for all } \mathbb{R} \in \wp, \omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i,j\}),$$

where $d(\cdot,\cdot)$ has the property given in Theorem 2.3.

For illustrative purpose, we consider the following example.

Example 3.4. Let $N = \{1,2,3,4\}$. For the purpose of comparison, in each of the following social structures, I, II, III, IV and V, we assume that (a) for all $i,j \in N$, there exists a benign chain from i to j; and (b) there exist no $i,j \in N$ such that i and j are hostile:

I. $1R2, 1R3, 1R4, 2R3, 2R4$ and $3R4$;

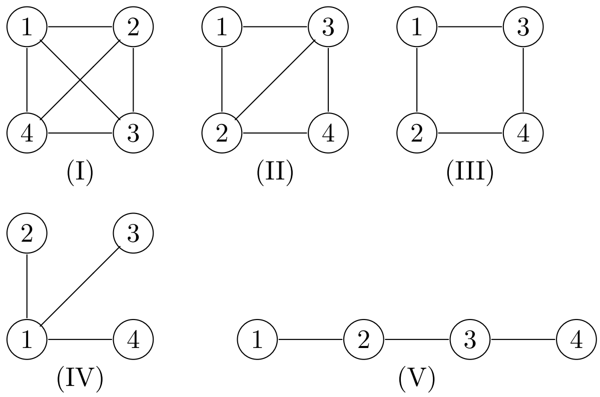
II. $1R2, 1R3, 2R3, 2R4$ and $3R4$;

III. $1R2, 1R3, 2R4$ and $3R4$;

IV. $1R2, 1R3$ and $1R4$;

V. $1R2, 2R3, 3R4$.

These five structures are illustrated in the following figure.



Structure V can be regarded as an *extreme hierarchy*, IV as a variant of hierarchies, III as a type of *corporations*, and II and I are variants of *horizontal* structure. For all $i, j \in N$, let t be the length of a shortest benign chain from i to j. Let $f(t)$ be the degree of personal connection between two individuals for t. Clearly, $f(1) \geq f(2) \geq f(3)$. For all $s \in \{I, II, III, IV, V\}$, let $\omega(s)$ be the extent of social networks under structure s. Then, we obtain the following:

$$\begin{aligned}\omega(I) &= 6f(1)\\ \omega(II) &= 5f(1) + f(2)\\ \omega(III) &= 4f(1) + 2f(2)\\ \omega(IV) &= 3f(1) + 3f(2)\\ \omega(V) &= 3f(1) + 2f(2) + f(3)\end{aligned}$$

It is then clear that

$$\omega(I) \geq \omega(II) \geq \omega(III) \geq \omega(IV) \geq \omega(V).$$

Thus, if one uses the extent of social networks to measure the amount of social capital in a society, then, in our example, the extreme hiearchy offers the least amount of social capital. This is in line with the findings of, for example, Putnam (1983) and La Porta, Lopez-de-Silanes, Shleifer, and Vishny (1997). ■

By Theorems 3.2 and 3.3, from Theorems 2.5, 2.6 and 2.7, the following results are immediate.

Corollary 3.5. Suppose Normalization and Assumption 2.4 are satisfied. Suppose further that d satisfies Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance, and ω satisfies Marginal Contritution and ω-Normalization.

3.5.1. If the process of message transmission is characterized by exogenously determined proportional message loss, where the fraction of message lost at each stage is given by ψ, then

$$\text{for all } \mathbb{R} \in \wp, \omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i, j\}),$$

where $d(\cdot, \cdot)$ has the property given in Theorem 2.5.

3.5.2. If the process of message transmission is characterized by endogenously determined message loss, then

$$\text{for all } \mathbb{R} \in \wp, \omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i, j\}),$$

where $d(\cdot, \cdot)$ has the property given in Theorem 2.7.

Corollary 3.6. Suppose Normalization and Assumption 2.4 are satisfied. Suppose further that d satisfies Simple Domination (I), Simple Weak Domination (II), Neutrality, Anonymity, Weak Independence, Monotonicity and Dominance, and ω satisfies Marginal Contritution and ω-Normalization.

3.6.1. If the process of message transmission is characterized by exogenously determined absolute message loss, where the absolute amount of message lost at each stage is given by α, then

$$\text{for all } \mathbb{R} \in \wp, \omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i, j\}),$$

where $d(\cdot, \cdot)$ has the property given in Theorem 2.6.

4. Concluding Remarks

In this paper, we have axiomatically developed measures of the personal connection between two individuals and also measures of the extent of social networks in a society. Our analysis suggests several directions for further exploration. First, in introducing the binary relation R (“having a good relation with”) over the set of all individuals, we did not distinguish the varying strengths of a (direct) good relation—“close friendship”, “ordinary friendship”, “a mildly friendly relation”, and so on. Intuitively, the strengths of the (direct) good relations involved in benign chains between two individuals would seem to be relevant in assessing the degree of (direct or indirect) personal connection between them. We have not discussed this issue in our paper. It is an issue that deserves separate investigation. Secondly, in discussing our axioms, we have referred to contexts where some of our axioms may not be very plausible. For example, each benign chain from p to q may serve as a valuable channel through which p can exert persuasive influence on q when the occasion requires it, and the benign chain may retain this value no matter how many other benign chains may be available. In that case, in assessing the degree of personal connection between p and q, it may not be possible to identify the effectiveness of a set of benign chains between p and q with the effectiveness of any single benign chain in the set. Also, in such cases, there may be possible tradeoffs between the range of benign chains available and the consideration of the lengths of these benign chains. We have not incorporated these aspects in our analysis; again, these aspects deserve separate investigation.

Appendix A

Proof of Theorem 2.2. It can be checked that if (1) holds for all $\{i,j\},\{p,q\} \in Z$ and all $\mathbb{R},\mathbb{R}' \in \wp$, then d satisfies Simple Domination (I), Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance. We now show that if d satisfies Simple Domination (I), Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance, then for all $\{i,j\},\{p,q\} \in Z$ and all $\mathbb{R},\mathbb{R}' \in \wp$, (1) holds.

Let d satisfy Simple Domination (I), Simple Domination (II), Neutrality, Anonymity, Independence, Monotonicity and Dominance. First, we note that, by Simple Domination (I), for all $\mathbb{R},\mathbb{R}' \in \wp$ and all $\{i,j\},\{p,q\} \in Z$,

> if i is isoloated from j under $\mathbb{R}$ and p is isolated from q under $\mathbb{R}'$, then
> $$d(\mathbb{R},\{i,j\}) = d(\mathbb{R}',\{p,q\}) \tag{2}$$

and

> if iRj and $pR'q$, then
> $$d(\mathbb{R},\{i,j\}) = d(\mathbb{R}',\{p,q\}). \tag{3}$$

Next, by Domination (II), for all $\mathbb{R},\mathbb{R}' \in \wp$ and all $\{i,j\},\{p,q\} \in Z$,

> if [iRj and not$pR'q$] or [i is linked with j under $\mathbb{R}$ and p is isolated from q under $\mathbb{R}'$, then
> $$d(\mathbb{R},\{i,j\}) > d(\mathbb{R}',\{p,q\}). \tag{4}$$

We now show that,

> **Claim 1**: For all $\mathbb{R}, \mathbb{R}' \in \wp$ such that $\mathbb{R}$ and $\mathbb{R}'$ are without hostility, for all $\{i, j\} \in Z$, if $(m(1) = i, m(2), \cdots, m(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}$ and $(m'(1) = i, m'(2), \cdots, m'(s+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}'$, then $d(\mathbb{R}, \{i, j\}) \geq d(\mathbb{R}', \{i, j\}) \Leftrightarrow t \leq s$.

Consider first that $s = t$. If $s = t = 1$, then, by (3), $d(\mathbb{R}, \{i, j\}) = d(\mathbb{R}', \{i, j\})$ follow immediately. Suppose $s = t > 1$. Then, by the successive use of Independence, we obtain $d(\mathbb{R}, \{i, j\}) \geq d(\mathbb{R}', \{i, j\})$ iff $d(\mathbb{R}, \{i, m(2)\}) \geq d(\mathbb{R}', \{i, m'(2)\})$. From (3), $d(\mathbb{R}, \{i, m(2)\}) = d(\mathbb{R}', \{i, m'(2)\})$. Therefore, $d(\mathbb{R}, \{i, j\}) = d(\mathbb{R}', \{i, j\})$ follows immediately. Consider now that $s > t$. Let $s = t + h$. It should be noted that for all $k = t+1, \cdots, s$, $(m'(1) = i, \cdots, m'(k))$ is the unique benign chain from i to $m'(k)$ under $\mathbb{R}'$. By the successive use of Independence, we obtain $d(\mathbb{R}, \{i, j\}) \geq d(\mathbb{R}', \{i, m'(t+2)\})$ iff $d(\mathbb{R}, \{i, m(2)\}) \geq d(\mathbb{R}', \{i, m'(3)\})$. From (4), clearly, $d(\mathbb{R}, \{i, m(2)\}) > d(\mathbb{R}', \{i, m'(3)\})$. Hence, $d(\mathbb{R}, \{i, j\}) > d(\mathbb{R}', \{i, m'(t+2)\})$. Similarly, we can show that $d(\mathbb{R}', \{i, m'(t+2)\}) > d(\mathbb{R}', \{i, m'(t+3)\}), \cdots, d(\mathbb{R}', \{i, m'(s)\}) > d(\mathbb{R}', \{i, m'(s+1) = j\})$. Therefore, $d(\mathbb{R}, \{i, j\}) > d(\mathbb{R}', \{i, j\})$. This completes the proof for Claim 1.

With Claim 1, we are now ready to show the following:

> **Claim 2**: For all $\mathbb{R}, \mathbb{R}' \in \wp$ and all $\{i, j\} \in Z$ such that $\mathbb{R}'$ is without hostility, if the length of a shortest benign chain from i to j under $\mathbb{R}$ is $t \geq 1$ and $m(1) = i, m(2), \cdots, m(t+1) = j$ is the unique benign chain from i to j under $\mathbb{R}'$, then $d(\mathbb{R}, \{i, j\}) = d(\mathbb{R}', \{i, j\})$.

Let $\{C_1, \cdots, C_h\}$ be the set of all benign chains from i to j under $\mathbb{R}$. For $k = 1, \cdots, h$, let t_k be the length of the benign chain in C_k. Without loss of generality, let $t = t_1 \leq t_2 \leq \cdots \leq t_h$. Consider $\mathbb{R}_1, \cdots, \mathbb{R}_h \in \wp$ such that, each and every one of $\mathbb{R}_1, \cdots, \mathbb{R}_h$ is without hostility and for all $k = 1, \cdots, h$, C_k is the unique benign chain from i to j under $\mathbb{R}_k$. From Claim 1 and the construction of $\mathbb{R}_1, \cdots, \mathbb{R}_h$, noting that $t_1 \leq \cdots \leq t_h$, we have

$$d(\mathbb{R}_1, \{i, j\}) \geq d(\mathbb{R}_2, \{i, j\}) \geq \cdots \geq d(\mathbb{R}_h, \{i, j\}). \quad (5)$$

For $g = 2, \cdots, h$, let $\mathbb{R}^g$ be such that the set of all benign chains from i to j under $\mathbb{R}^g$ is $\{C_1, \cdots, C_g\}$. Clearly, the set of all benign chains from i to j under $\mathbb{R}^h$ is $\{C_1, \cdots, C_h\}$, which is the same as the set of all benign chains from i to j under $\mathbb{R}$. By Neutrality, therefore,

$$d(\mathbb{R}, \{i, j\}) = d(\mathbb{R}_h, \{i, j\}). \quad (6)$$

Then, by Dominance, noting that the set of all benign chains from i to j under $\mathbb{R}^2$ is the union of the sets of all benign chains from i to j under $\mathbb{R}_1$ and under $\mathbb{R}_2$ and $d(\mathbb{R}_1, \{i, j\}) \geq d(\mathbb{R}_2, \{i, j\})$, we obtain

$$d(\mathbb{R}_1, \{i, j\}) \geq d(\mathbb{R}^2, \{i, j\}).$$

Similarly, by Dominance, noting that the set of all benign chains from i to j under $\mathbb{R}^3$ is the union of the sets of all benign chains from i to j under $\mathbb{R}^2$ and under $\mathbb{R}_3$ and $d(\mathbb{R}_1, \{i,j\}) \geq d(\mathbb{R}^2, \{i,j\}) \geq d(\mathbb{R}_3, \{i,j\})$, we obtain

$$d(\mathbb{R}_1, \{i,j\}) \geq d(\mathbb{R}^2, \{i,j\}) \geq d(\mathbb{R}^3, \{i,j\}).$$

By the repeated use of the above, from Dominance, we obtain

$$d(\mathbb{R}_1, \{i,j\}) \geq d(\mathbb{R}^2, \{i,j\}) \geq \cdots \geq d(\mathbb{R}^h, \{i,j\})$$

By Monotonicity, however, we have

$$d(\mathbb{R}^h, \{i,j\}) \geq d(\mathbb{R}_1, \{i,j\}).$$

Therefore, $d(\mathbb{R}^h, \{i,j\}) = d(\mathbb{R}_1, \{i,j\})$. (6) now imlies

$$d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}_1, \{i,j\}). \tag{7}$$

On the other hand, from Claim 1, $d(\mathbb{R}', \{i,j\}) = d(\mathbb{R}_1, \{i,j\})$. Therefore, $d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}', \{i,j\})$. Note that the length of a shortest benign chain from i to j under $\mathbb{R}$ is $t_1 = t$, which is the length of the unique benign chain from i to j under $\mathbb{R}'$. Thus, Claim 2 is proved.

By Claims 1 and 2, we obtain the following:

> **Claim 3**: For all $\mathbb{R}, \mathbb{R}' \in \wp$ and all $\{i,j\} \in Z$, if t is the length of a shortest benign chain from i to j under $\mathbb{R}$ and s is the length of a shortest benign chain from i to j under $\mathbb{R}'$, then $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\}) \Leftrightarrow t \leq s$.

To see that Claim 3 is true, consider $\mathbb{R}_1, \mathbb{R}_2 \in \wp$ such that both $\mathbb{R}_1, \mathbb{R}_2$ are without hostility, $(m(1) = i, m(2), \cdots, m(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}_1$ and $(m'(1) = i, m'(2), \cdots, m'(s+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}_2$. By Claim 1, $d(\mathbb{R}_1, \{i,j\}) \geq d(\mathbb{R}_2, \{i,j\}) \Leftrightarrow t \leq s$. By Claim 2, $d(\mathbb{R}_1, \{i,j\}) = d(\mathbb{R}, \{i,j\})$ and $d(\mathbb{R}_2, \{i,j\}) = d(\mathbb{R}', \{i,j\})$. Therefore, $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{i,j\})$ iff $t \leq s$.

Finally, we are ready to show (1). Let $\mathbb{R}, \mathbb{R}' \in \wp$ and $\{i,j\}, \{p,q\} \in Z$. If i and j are isolated under $\mathbb{R}$, and p and q are isolated under $\mathbb{R}'$, then from (2), $d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}', \{p,q\})$. If i is linked with j under $\mathbb{R}$, and p and q are isolated under $\mathbb{R}'$, then by (4), $d(\mathbb{R}, \{i,j\}) > d(\mathbb{R}', \{p,q\})$. Suppose now that i and j are linked under $\mathbb{R}$, and p and q are linked under $\mathbb{R}'$. Let $(m(1) = i, m(2), \cdots, m(t+1) = j)$ be a shortest benign chain from i to j under $\mathbb{R}$, and $(m'(1) = p, m'(2), \cdots, m'(s+1) = q)$ be a shortest benign chain from p to q under $\mathbb{R}'$. If $\{i,j\} = \{p,q\}$, then, by Claim 3, (1) follows immediately. We therefore consider two cases: ($i = p$ and $j \neq q$), and ($\{i,j\} \cap \{p,q\} = \emptyset$). The cases in which ($i = q$ and $j \neq p$), ($j = p$ and $i \neq q$), and ($j = q$ and $i \neq p$) are similar to the case in which ($i = p$ and $j \neq q$). Consider first that ($i = p$ and $j \neq q$). Consider $\mathbb{R}_1$, $\mathbb{R}_2 \in \wp$ such that: $(m(1) = i, m(2), \cdots, m(t) = q, m(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}_1$, and $(m'(1) = p, m'(2), \cdots, m'(s) = j, m'(s+1) = q)$ is the unique benign chain from p to q under $\mathbb{R}_2$. By Claim 3,

$$d(\mathbb{R}, \{i,j\}) = d(\mathbb{R}_1, \{i,j\}) \text{ and } d(\mathbb{R}', \{p,q\}) = d(\mathbb{R}_2, \{p,q\}). \tag{8}$$

Consider the one-to-one function σ from N to N such that: for all $k \in N - \{j,q\}, \sigma(k) = k$ and $\sigma(j) = q, \sigma(q) = j$. Let $\mathbb{R}_3 \in \wp$ be such that, for all $k,h \in N, kR_2h$ iff $\sigma(k)R_3\sigma(h)$, and $kR_2(-)h$ iff $\sigma(k)R_3(-)\sigma(h)$. It is clear that $(\sigma(m'(1)) = i, \sigma(m'(2)), \cdots, \sigma(m'(s)) = \sigma(j) = q, \sigma(m'(s+1)) = \sigma(q) = j)$ be the unique benign chain from i to j under $\mathbb{R}_3$. By Claim 3, we have:

$$d(\mathbb{R}_1, \{i,j\}) \geq d(\mathbb{R}_3, \{i,j\}) \Leftrightarrow t \leq s. \tag{9}$$

And by Anonymity, we obtain

$$d(\mathbb{R}_2, \{p,q\}) = d(\mathbb{R}_3, \{i,j\}). \tag{10}$$

Therefore, from (10), (9) and (8), we obtain $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{p,q\}) \Leftrightarrow t \leq s$. Hence, (1) holds in this case. When $\{i,j\} \cap \{p,q\} = \emptyset$, consider the one-to-one function σ' from N to N such that for all $k \in N - \{i,j,p,q\}$, $\sigma'(k) = k$, $\sigma'(i) = p$ and $\sigma'(j) = q$. Let $\mathbb{R}_4 \in \wp$ be such that for all $k,h \in N$, $kR'h$ iff $\sigma'(k)R_4\sigma'(h)$ and $kR'(-)h$ iff $\sigma'(k)R_4(-)\sigma'(h)$. Then, by Anonymity,

$$d(\mathbb{R}', \{p,q\}) = d(\mathbb{R}_4, \{i,j\}). \tag{11}$$

By Claim 3,

$$d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}_4, \{i,j\}) \Leftrightarrow t \leq s. \tag{12}$$

By (11) and (12), we obtain $d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{p,q\}) \Leftrightarrow t \leq s$. Therefore, (1) holds for this case. ■

Proof of Theorem 2.3. The proof is similar to that of Theorem 2.2 and we omit it. ■

Appendix B

Proof of Theorem 2.5. The "if" part of the theorem can be checked. We now prove the "only if" part. Let $\psi \in (0,1)$ and the process of message transmission be characterized by exogenously determined proportional message loss. Let d satisfy the axioms specified in Theorem 2.5. From Theorem 2.2, we need only to show that, for all $\mathbb{R} \in \wp$ and all $\{i,j\} \in Z$, if $C = (m(1) = i, m(2), \cdots, m(t+2) = j)$ is the unique benign chain from i to j under $\mathbb{R}$ with $t \geq 1$, then $d(\mathbb{R}, \{i,j\}) = (1-\psi)^t$. Since the process of message transmission is characterized by exogenously determined proportional message loss, we have the following:
$a_{m(2)}[C] = 1, a^L_{m(2)} = 1 \cdot \psi$;
$a_{m(3)}[C] = 1 - \psi, a^L_{m(3)}[C] = \psi(1-\psi)$;
$\cdots$;

$a_{m(t+1)}[C] = (1-\psi)^{t-1}, a^L_{m(t+1)}[C] = \psi(1-\psi)^{t-1}$;
$a_{m(t+2)}[C] = (1-\psi)^{t-1} - \psi(1-\psi)^{t-1} = (1-\psi)^t$.

Therefore, $d(\mathbb{R}, \{i,j\}) = (1-\psi)^t$. ■

Proof of Theorem 2.6. The "if" part of the theorem can be checked. We now prove the "only if" part. Let $\alpha > 0$ and the process of message transmission be characterized by exogenously determined absolute message loss. Let d satisfy the axioms specified in Theorem 2.6. From Theorem 2.3, we need only to show that, for $\mathbb{R} \in \wp$ and all $\{i,j\} \in Z$, if $C = (m(1) = i, m(2), \cdots, m(t+2) = j)$ is the unique benign chain from i to j under $\mathbb{R}$ with $t \geq 1$, then $d(\mathbb{R}, \{i,j\}) = \max\{1-\alpha t, 0\}$. By Assumption 2.4, $d(\mathbb{R}, \{i,j\}) = a_{m(t+2)}[C]$. Since the process of message transmission is characterized by exgogenously determined absolute message loss, we have
$a_{m(2)}[C] = 1, a^L_{m(2)} = \alpha$;
$a_{m(3)}[C] = \max\{1-\alpha, 0\}, a^L_{m(3)}[C] = \max\{\min\{a_{m(3)}[C] - \alpha, \alpha\}m, 0\}$;
$\cdots$;
$a_{m(t+1)}[C] = \max\{1-\alpha(t-1), 0\}, a^L_{m(t+1)}[C] = \max\{\min\{a_{m(t+1)}[C]-\alpha, \alpha\}, 0\}$;
$a_{m(t+2)}[C] = a^L_{m(t+1)}[C] = \max\{a_{m(t+1)}[C] - \alpha, 0\} = \max\{1-\alpha t, 0\}$.

Therefore, $d(\mathbb{R}, \{i,j\}) = \max\{1 - t\alpha, 0\}$. ■

Proof of Theorem 2.7. The "if" part of the theorem can be checked. We now prove the "only if" part. Let d satisfy the axioms specified in Theorem 2.7. Suppose the process of message transmission is characterized by endogenously determined message loss. By Theorem 2.2, we need only to show that, for all $\mathbb{R} \in \wp$ and all $\{i,j\} \in Z$, if $C = (m(1) = i, \cdots, m(t+1) = j)$ is the unique benign chain from i to j under $\mathbb{R}$ where $t > 1$, then $d(\mathbb{R}, \{i,j\}) = 1/t$. By Assumption 2.4, $d(\mathbb{R}, \{i,j\}) = a_{m(t+1)}[C]$. Since the process of message transmission is characterized by endogenously determined message loss, we have
$a_{m(t+1)}[C] = a^L_{m(t)}[C] = a_{m(t)}[C]$,
$a_{m(t)}[C] + a^L_{m(t)}[C] = a_{m(t-1)}[C]$,
$a^L_{m(t-1)}[C] = a_{m(t-1)}[C]$,
$a_{m(t-1)}[C] + a^L_{m(t-1)}[C] = a_{m(t-2)}[C]$,
$\cdots$,
$a_{m(2)}[C] = a^L_{m(2)}[C]$
$a_{m(2)} + a^L_{m(2)} = 1$.

Therefore, $d(\mathbb{R}, \{i,j\}) = 1/t$. ■

Appendix C

Proof of Theorem 3.2. Suppose d satisfies the axioms specified in Theorem 3.2, and ω satisfies Marginal Contritution and ω-Normalization. Then, by Theorem 2.1, for all $\{i,j\}, \{p,q\} \in Z$, all $\mathbb{R}, \mathbb{R}' \in \wp$,

$$d(\mathbb{R}, \{i,j\}) \geq d(\mathbb{R}', \{p,q\}) \text{ iff } ([t \leq s] \text{ or } [p \text{ is isolated from } q \text{ under } \mathbb{R}'])$$

where t is the length of a shortest benign chain from i to j under $\mathbb{R}$ and s is the length of a shortest benign chain from p to q under $\mathbb{R}'$.
Let $\mathbb{R} \in \wp$ be given. If for all $\{i', j'\} \in Z$, not$(i'Rj')$, then, by ω-Normalization, $\omega(\mathbb{R}) = 0$. Assume, therefore, that $i'Rj'$ for some $\{i', j'\} \in Z$ under $\mathbb{R}$. For all $\{i, j\} \in Z$ and all $\mathbb{R}' \in \wp$, let $s(\mathbb{R}', ij)$ be the length of a shortest benign chain from i to j under $\mathbb{R}'$. Let $\{i^1, j^1\} \in Z$ be such that $s(\mathbb{R}, i^1 j^1) \geq s(\mathbb{R}, ij)$ for all $\{i, j\} \in Z$. Consider $\mathbb{R}_1$ such that: for all $\{p, q\} \in Z$ with $\{p, q\} \neq \{i^1, j^1\}$, $[(pRq \Leftrightarrow pR_1q)$ and $(pR(-)q \Leftrightarrow pR_1(-)q)]$, and $i^1R_1(-)j^1$. Given that there is at least one benign chain from i^1 to j^1 under $\mathbb{R}$, clearly, not$(i^1R(-)j^1)$. Since $\{i^1, j^1\}$ is such that $s(\mathbb{R}, i^1j^1) \geq s(\mathbb{R}, ij)$ for all $\{i, j\} \in Z$, for all $\{p, q\} \in Z$ with $\{i^1, j^1\} \neq \{p, q\}$, a shortest benign chain from p to q under $\mathbb{R}$ does not go through i^1 and j^1. Therefore, when the only change from $\mathbb{R}$ to $\mathbb{R}_1$ involving switching not$(iR(-)j)$ to $iR_1(-)j$, we must have: for all $\{p, q\} \in Z$ with $\{p, q\} \neq \{i^1, j^1\}$, $[(pRq \Leftrightarrow pR_1q)$ and $(pR(-)q \Leftrightarrow pR_1(-)q)]$, and the set of all shortest benign chains from p to q under $\mathbb{R}$ is the set of all shortest benign chains from p to q under $\mathbb{R}_1$. It is then clear that $\mathbb{R}_1 \in \wp$ and $d(\mathbb{R}, \{p, q\}) = d(\mathbb{R}_1, \{p, q\})$ for all $\{p, q\} \in Z$ with $\{p, q\} \neq \{i^1, j^1\}$. By Marginal Contribution, we must have

$$\omega(\mathbb{R}) - \omega(\mathbb{R}_1) = d(\mathbb{R}, \{i^1, j^1\}). \tag{13}$$

Clearly, there is no benign chain from i^1 to j^1 under $\mathbb{R}_1$. If for all $\{i, j\} \in Z$, there is no benign chain from i to j under $\mathbb{R}_1$, then by ω-Normalization, $\omega(\mathbb{R}_1) = 0$, and the conclusion of Theorem 4.2 follows easily from (13). Let $\{i^2, j^2\} \in Z$ be such that $s(\mathbb{R}_1, i^2j^2) \geq s(\mathbb{R}_1, ij)$ for all $\{i, j\} \in Z$. Consider $\mathbb{R}_2$ such that: for all $\{p, q\} \in Z$ with $\{p, q\} \neq \{i^2, j^2\}$, $[(pR_1q \Leftrightarrow pR_2q)$ and $(pR_1(-)q \Leftrightarrow pR_2(-)q)]$, and $i^2R_2(-)j^2$. Following a similar argument for $\mathbb{R}$ and $\mathbb{R}_1$, we can show that $\mathbb{R}_2 \in \wp$ and $d(\mathbb{R}_1, \{p, q\}) = d(\mathbb{R}_2, \{p, q\})$ for all $\{p, q\} \in Z$ with $\{p, q\} \neq \{i^2, j^2\}$. By Marginal Contribution again, we obtain

$$\omega(\mathbb{R}_1) - \omega(\mathbb{R}_2) = d(\mathbb{R}_1, \{i^2, j^2\}). \tag{14}$$

Note that $d(\mathbb{R}_1, \{i^2, j^2\}) = d(\mathbb{R}, \{i^2, j^2\})$. Therefore,

$$\omega(\mathbb{R}) = \omega(\mathbb{R}_2) + d(\mathbb{R}, \{i^1, j^1\}) + d(\mathbb{R}, \{i^2, j^2\}). \tag{15}$$

By repeating the above procedures and from the repeated use of Marginal Contribution, since Z contains a finite number of elements, we can obtain

$$\omega(\mathbb{R}) = \sum_{\{i,j\} \in Z} d(\mathbb{R}, \{i, j\}).$$

■

Proof of Theorem 3.3. The proof is similar to that of Theorem 3.2 and we omit it. ■

Bibliography

Bowles, S./H. Gintis (2002), Social Capital and Community Governance, in: *Economic Journal 112*, F419–F436

Coleman, J. S. (1990), *Foundations of Social Theory*, Cambridge

— (2000), Social Capital in the Creation of Human Capital, in: P. Dasgupta/I. Serageldin (eds.), *Social Capital*, Washington

Dasgupta, P. (2000), Economic Progress and the Idea of Social Capital, in: P. Dasgupta/I. Serageldin (eds.), *Social Capital*, Washington

Durlauf, S. N. (2002), On the Empirics of Social Capital, *Economic Journal 112*, F459–479

Dutta, B./M. O. Jackson (2001) On the Formation of Networks and Groups, forthcoming in: B. Dutta/M. O. Jackson (eds.), *Models of the Strategic Formation of Networks and Groups*

Glaeser, E. L./D. Laibson/B. Sacerdote (2002), An Economic Approach to Social Capital, in: *Economic Journal 112*, F437–458

—/—/D. Scheinkman/C. Soutter (2000), Measuring Trust, in: *Quarterly Journal of Economics 115*, 811–846

Knack, S./P. Keefer (1997), Does Social Capital Have an Economic Pay-off? A Cross-country Investigation, in: *Quarterly Journal of Economics 112*, 1251–1288

La Porta, R./F. Lopez-de-Silanes/A. Shleifer/R. W. Vishny (1997), Trust in Large Organizations, in: *American Economic Review: Papers and Proceedings 87*, 333–338

Marsden. P. V. (1990), Network Data and Measurement, in: *Annual Review of Sociology 16*, 435–463

Moreno, J. L. (1934), *Who Shall Survive? Foundations of Sociometry, Group Psychotherapy, and Sociodrama*, Washington

Putnam, R. D. (1993), *Making Democracy Work: Civic Traditions in Modern Italy*, Priceton

— (1995), Bowling Alone: America's Declining Social Capital, in: *Journal of Democracy 6*, 65–78

Scott, J. (1991), *Social Network Analysis: A Handbook*, Thousand Oaks: Sage

Stiglitz, J. E. (2000), Formal and Informal Institutions, in: P. Dasgupta/I. Serageldin (eds.), *Social Capital*, Washington

Wasserman, S/K. Faust (1994), *Social Network Analysis*, New York