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On Chaotic Consistent Expectations Equilibria

Abstract: The notion of consistent expectations equilibria (CEE) as propagated by Hommes/Sorger (1998) is reviewed. Focusing on their example of a chaotic CEE constructed in the context of a cobweb model, it is argued that such an equilibrium is a temporary one. Assuming that an agent—modeled as an individual, versatile in applying the basic tools of *linear* time-series econometrics—has learned the CEE, I analyze the duration of the time period over which the agent maintains her/his beliefs concerning the perceived law of motion (AR(1)). The analysis based on numerical simulations indicates that the use of techniques rooted in the *linear* paradigm is sufficient to generate convincing evidence against the underlying perceived law of motion.

0. Introduction

Economic systems today are typically viewed as highly self-referential systems. The perception of the system held by agents will determine the actual nature of the system, while the nature of the system will, in turn, have a profound effect on the agents' understanding of the system. There exists a strand of literature in economic theory which conceptualizes this apparent quality of self-reference in the form of expectation-feedback systems. Expectations held by subjects in the economy concerning key variables determine the actual dynamics of the economic system, which then feeds back into the formation of expectations. The explicit dynamic modeling of such feedback systems poses a multitude of challenges along its technical as well as conceptual dimension. Contributions to this area of research with an emphasis on the stability properties of adaptive learning include those of Marcet/Sargent (1989b), Marcet/Sargent (1989a), Bullard (1994) and Schoenhofer (1999). Evans/Honkapohja (1994), Evans/Honkapohja (1995) focus on expectational stability of adaptive learning rules. Kurz (1994) introduces rational belief equilibria. Böhm/Wenzelburger (1999) propose a dynamical systems perspective and introduce the concept of perfect predictors.

A recent example for a research effort focusing on the interplay between expectation formation and nonlinear actual dynamics of an economic system is the paper by Hommes/Sorger (1998). The authors introduce the concept of a consistent expectation equilibrium (CEE) in connection with nonlinear economic models. Their equilibrium concept incorporates aspects of Sargent's notion of *bounded rationality* with a certain consistency requirement. Following the strategy of the Sargent-type bounded rationality program, informational requirements are reduced by assuming that economic agents have all the characteristics which allow them "to behave like working economists or econometricians" (Sargent 1993, 22). The assumption that the decision makers in a model are omniscient in the sense of possessing perfect insight into the structure of the economy is substituted by a premise concerning their mental capacity. This intellectual endowment is sufficient to let the individual perform inferential procedures on the basis of information provided in the form of data. By assuming that the inference makers have been educated under and "locked into" a *linear statistical paradigm*, Hommes and Sorger introduce a restriction on the type of econometrician considered. The consistency requirement being the central, defining feature of the CEE relates the agent's perception of the process generating an economic variable to the actual observable moments of this variable. The original formulation of the equilibrium concept involves the first two central moments of the variable of interest. Both the *perceived law* and the *actual law of motion* are deterministic. An adaptation of the CEE to a probabilistic setting has been proposed by Crespo-Cuaresma/Sorger (1999). Recently, Sögner/Mitlöhner (2002) employ the equilibrium concept in a capital market context. They show that the only CEE in their stock market model is the rational expectations equilibrium.

In their original contribution concerning consistent expectations equilibria, Hommes and Sorger demonstrate the existence of a chaotic CEE in the framework of a cobweb model. Initially believing that an autoregressive process of order one (AR(1)) represents an adequate abstraction for the price dynamics, agents enter into a learning process based on price data which are generated by a nonlinear map. The *learning activity*, modeled as an updating process on the parameters of the *perceived linear law of motion*, which also parametrize the *actual nonlinear law of motion* for the prices, converges to a point in the parameter space associated with a stable AR(1) process. So eventually, the agent believes in her perceived law of motion although the prices are generated by a chaotic map.

The purpose of this note is to demonstrate that, given the assumptions of the model, the fixed point of the expectations-feedback system which is identified as a chaotic CEE represents a *temporary phenomenon.* This goal is achieved by performing two types of experiments. Each combines artificial data generated in the chaotic CEE with the simulated use of statistical tools typically associated with *linear* time-series statistics. In the first experiment, I simulate the activities of a hypothetical statistician familiar with basic concepts of linear time-series analysis. Confronted with a batch of price data generated by a chaotic CEE, her statistical analysis relies on the concept of autocorrelation. It is concluded that this type of agent, who is boundedly rational in the sense of Sargent, will be able to cast significant doubt concerning her perceived law of motion. The second experiment takes the form of a numerical simulation experiment. An individual is modeled who observes possibly noisy data under the conditions of the chaotic CEE. Apart from being familiar with basic concepts of statistical inference, she has access to a well-known specification test (Ramsey's RESET test). This testing technology basically enables the individual to gather evidence against her perceived law of motion. On the basis of simulated decisions of such an agent, estimates of the time elapsed until the perceived law is rejected are computed.

The results clearly suggest that even in the case of noisy data the chaotic CEE propagated by Hommes and Sorger will be a temporary phenomenon.

The paper is organized as follows: A short formal account of the situation referred to as the chaotic CEE by Hommes & Sorger is given in Section 1. The designs as well as the outcomes of the two experiments are described in the following sections. The case of the statistician employing the autocorrelation-function tool in the presence of data generated in a chaotic CEE is considered in Section 2. In Section 3, the numerical simulations which mimic the use of specification testing are discussed. A summary together with implications for possible future research efforts is provided in Section 4. An appendix including a short account of the specification testing technology employed in the simulation exercise in Section 3 constitutes the final section of the paper.

1. The Chaotic CEE

The experiments described below rest on the premise that the economic system under consideration has converged to a state in which the defining conditions of a chaotic CEE prevail. In particular, I assume the scenario which is used in Hommes/Sorger (1998, 314). An agent who holds an AR(1) hypothesis concerning the price process in the context of a nonmonotonic *cobweb model*, has "learned" to believe in a specific autoregressive process. Learning is modeled as a data-driven online-estimation process on the parameter space of the family of linear autoregressive processes of order one. By relying on a variant of the least-squares updating algorithm, Hommes/Sorger (1998) specify a learning process which they refer to as the *sample autocorrelation* (SAC) learning process. Given the assumptions underlying their cobweb model, Hommes and Sorger demonstrate that while the agent perceives the law of motion to be of low order and linear, the actual price dynamics is non-linear and chaotic. In particular, the authors show that the evolution of their self-referential cobweb model can be described by the dynamical system

$$
\begin{aligned}
\alpha_t &= \frac{1}{t+1}\sum_{j=0}^{t} p_j \\
\beta_t &= \frac{\sum_{j=0}^{t-1}(p_j-\alpha_t)(p_{j+1}-\alpha_t)}{\sum_{j=0}^{t}(p_j-\alpha_t)^2} \\
p_{t+1} &= F(\alpha_t+\beta_t(p_t-\alpha_t))
\end{aligned}
$$

with

$$
F(p)=\begin{cases} 9-\frac{25}{2}\,p & \text{if } 0\le p\le \frac{18}{25} \\ \frac{25}{18}\,p-1 & \text{if } \frac{18}{25}\le p\le \frac{18}{5} \\ 4-\epsilon\,[\,p-\frac{18}{5}\,] & \text{if } \frac{18}{5}\le p\le \frac{18}{5}+\frac{4}{\epsilon} \\ 0 & \text{if } p\ge \frac{18}{5}+\frac{4}{\epsilon} \end{cases}
$$

where p_t denotes the price at time t and α_t and β_t refer to the value of the parameters in the perceived linear law at time t. In Hommes/Sorger (1998, 314) the authors point out that for $\epsilon = 0.25$ the system converges to $(\alpha, \beta) = (2.22, -0.94)$. Thereby they provide an example for convergence to the CEE. Although confronted with price data generated by a nonlinear law, the individual who is assumed to behave like a statistician believes in one member of the family of linear autoregressive processes of order one. A sample from the (price-) data-generating process associated with this CEE is exhibited in Figure 1. In the sequel of the paper, we will not be concerned with the convergence to the CEE. It is the situation prevailing after convergence has occurred which is of interest. In essence, I focus on the question of what type of inference an individual who is assumed to have the mental capacity to behave like a statistician will produce on the basis of the highly irregular price data exhibited in the figure below.

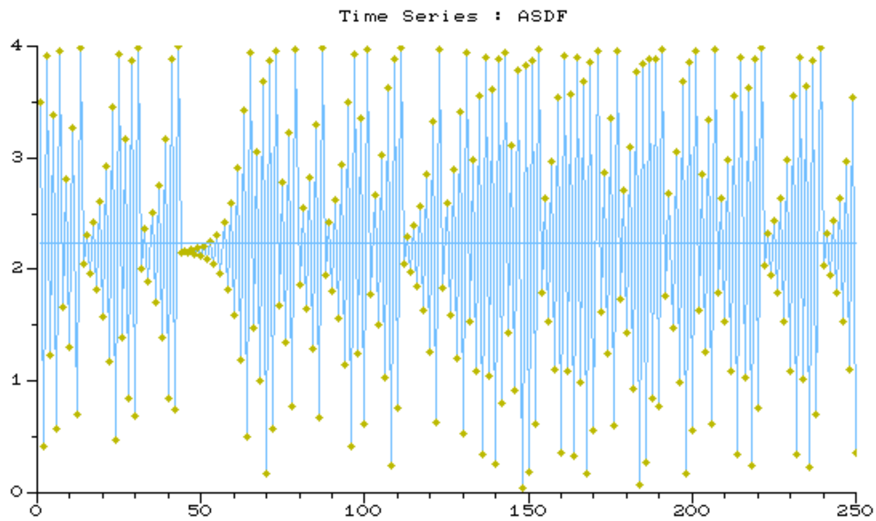

Figure 1: Price trajectory $\{p_t\}_0^{250}$ generated in the CEE ($p_0 = 3.5$)

2. CEE and Autocorrelation

Suppose the agent holding the AR(1) beliefs is endowed with basic skills in linear time-series analysis. Having been familiarized with Box-Jenkins type methodology, our econometrician has reached an understanding of the concept of an autocorrelation function.
Perceiving the law of motion for the price to equal

$$p_t = \alpha + \beta\,(p_{t-1} - \alpha) + \epsilon_t \tag{1}$$

she is aware of the fact that the linear properties of the price process can be described in terms of the moments $E[p_t]$ and

$$\lambda_\tau = cov[p_t, p_{t-\tau}] = E[\,(\,p_t - E[p_t]\,)(\,p_{t-\tau} - E[p_{t-\tau}]\,)\,] \tag{2}$$

for $\tau = 0, 1, 2, \ldots$. Moreover, the individual has understood that on the basis of the covariances one can devise the autocorrelation function

$$\rho_\tau = \frac{\lambda_\tau}{\lambda_0} \tag{3}$$

as a tool suitable to represent the extent of the linear relationship prevailing between the price at time t and the price at time $t - \tau$. Considering her AR(1) beliefs, the autocovariances take the form

$$\lambda_\tau = \beta^\tau \lambda_0 \quad \forall\, \tau \tag{4}$$

and consequently she would expect the strength and length of the memory of the price process being described adequately by the autocorrelation function

$$\rho_\tau = \beta^\tau, \quad \forall\, \tau. \tag{5}$$

Believing that the price process (1) with $(\alpha, \beta) = (2.22, -0.94)$ is an adequate description of reality is equivalent to describing the memory properties of the price process in terms of the autorcorrelation function as

$$\rho_\tau = (-0.94)^\tau \quad \forall\, \tau. \tag{6}$$

Now let us assume that the CEE has prevailed for T periods during which some agency observed, recorded and published the price data. This batch of data is available to the agent. One should recall at this point that the CEE has been reached under the SAC learning procedure. Agents capable of performing such a process of sequential *estimation* should be able to carry out the other type of inference, namely hypothesis *testing*. In this activity of contrasting hypotheses, e.g. the *perceived law*, with observations on the variable of interest—here price data generated by the *actual economic law*—lies the essence of learning behavior.

Given the assumptions under which the CEE is devised, it is reasonable to expect the individual in the model to eventually ask the question whether or not her perception of the price process is matched by the reality as it is manifested in the observed price data. Even if we assume a low level of sophistication, the econometrician will be able to devise a sample analogon of the autocorrelation function, i.e. a correlogram. In basic courses on time-series analysis, estimates of the autocorrelation coefficients are typically motivated by estimating the linear relationship between two elements of the process under scrutiny separated by lag τ. Suppose our econometrician adopts this strategy. She plots p_t versus $p_{t-\tau}$ and—since she is caught in the linear paradigm—she poses a linear relationship and computes the slope parameter of the associated linear regression model.

Plotting p_t vs. p_{t-1}, as done in Figure 2, reveals what the modeler already knows: the actual nonlinear law of motion of prices. The thin line superimposed on the scatter plot represents the estimated regression $p_t = \hat{\gamma}_0 + \hat{\gamma}_1\, p_{(t-1)}$. where

$(\hat{\gamma}_0, \hat{\gamma}_1)$ denotes the simple OLS estimate of the regression coefficients. This slope coefficient $\hat{\gamma}_1 = -0.9219$ constitutes an estimate of the autocorrelation coefficient ρ_1. The estimated standard error for the OLS estimate $\hat{\gamma}_1$ amounts to $\hat{\sigma}_{\hat{\gamma}_1} = 0.0249$. It implies a significant slope (autocorrelation) estimate. An estimate for ρ_2 can be devised in an analogous manner. Estimating $p_t = \hat{\gamma}_0 + \hat{\gamma}_1 \, p_{(t-2)}$ on the basis of the CEE prices plotted below, yields an estimate for $\rho_2 = .8496$. Selected scatter plots with the estimated regression lines superimposed are given in Figures 3—7. The estimates $\hat{\rho}_\tau$ for $\tau = 1, 2, \ldots 10$ along with their respective standard deviations are presented in columns 3 and 4 of Table 1. In the second column of that table, the values of the autocorrelation coefficients are listed which will be expected by an agent who believes in an AR(1) price process with $(\alpha, \beta) = (2.22, -0.94)$. Scrutiny of columns 2 and 3 reveals that the autocorrelation coefficients implied by the AR(1) beliefs decrease faster than the sample autocorrelation coefficients $\hat{\rho}_\tau$. The sample magnitudes do not show the same exponential decay present in population magnitudes ρ_τ.

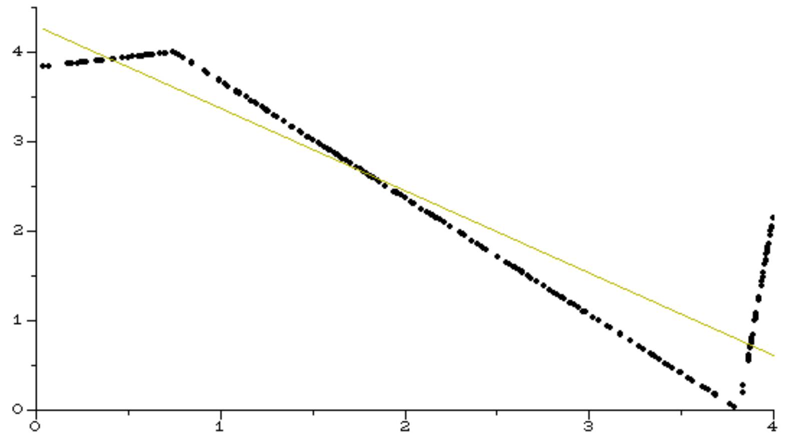

Figure 2: p_t vs. p_{t-1} in the CEE

Suppose the agent facing the apparent discrepancy in memory properties extends the analysis to even higher lags. The result is summarized in Figure 8.

Considering higher order lags, the individual finds that the exponential decay in the ρ's she expected on the grounds of her AR(1) beliefs is not a feature of the observed price process. There is too much memory in the observed process. Given the assumed capacity of the agent, the existing evidence will generate doubts against her initial hypothesis (*perceived law of motion*). It should be emphasized that the maintained belief has to be corrected on the grounds of evidence embodied in a correlogram. Given the econometricians perception concerning the dynamics of the price process, her choice of this tool is adequate.

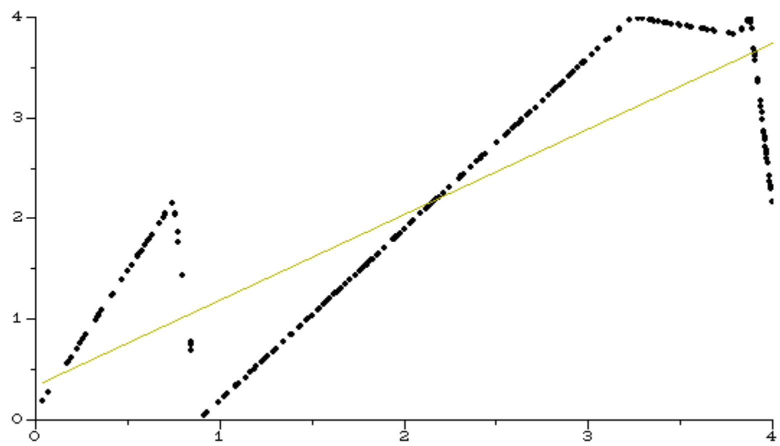

Figure 3: p_t vs. p_{t-2} in the CEE

After all, the autocorrelation function is the main tool for identification exercises in linear dynamical systems.

We conclude that an individual who behaves like an econometrician will eventually accumulate evidence against her initial hypothesis by using tools rooted in the linear paradigm. In the language of the CEE concept, this implies a variation of the *perceived law of motion*. From a formal point of view, then the dynamical system (1)-(3) is no longer an adequate description of the situation. A change in the perceived law of motion for the price, for example, to a higher order linear AR(p) or even some nonlinear process, implies an alternative prediction rule for the future price. As a consequence, the dynamical system (1)–(3) is no longer a valid description of the dynamic process. The specific linear predictor chosen by the agent fails to be self-confirming. In this sense, the learnable *chaotic CEE* of Hommes/Sorger (1998) constitutes a *temporary phenomenon*.

3. CEE and Specification Testing

The argument just presented rests on the assumption that the market participant in a cobweb model has command over a *specific* statistical technology. To counteract the possible argument that the temporary nature of the chaotic CEE hinges on the choice of a specific statistical tool, we present a simulation experiment. Again, in accordance with the Hommes and Sorger approach, the agent is modeled as a naive econometrician who is familiar with linear statistical techniques only. Contrary to the previous exercise, it is assumed that the individual is not familiar with Box-Jenkins technology. Instead, she is aware of

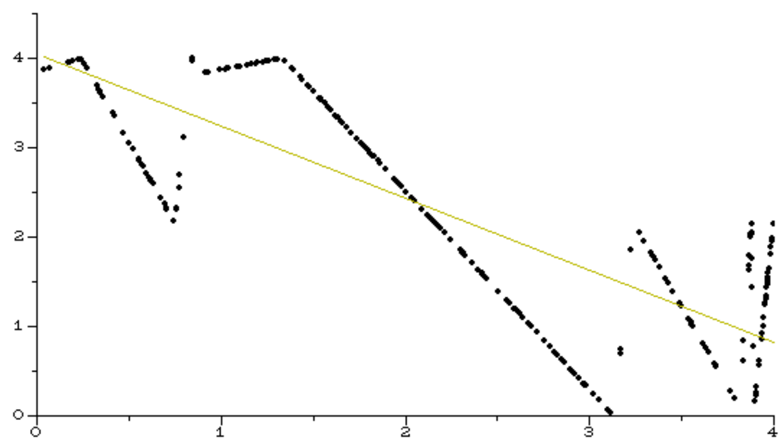

Figure 4: p_t vs. p_{t-3} in the CEE

basic concepts of linear regression. OLS estimation technology is available to her. In addition, we presume the existence of observation error in connection with the price data. The price information available at time t, $\tilde{p}_t$, is a mixture of a realization of the CEE price process p_t and a realization of an *i.i.d.* random variable ϵ_t

$$\tilde{p}_t = p_t + \epsilon_t. \tag{7}$$

With respect to observation error, it is assumed that $\epsilon_t \sim u[a, b]$ where $u[a, b]$ represents the family of uniform densities with parameters $a, b \in \mathbb{R}$. The parameters a and b will be varied subject to the constraint that $E[\epsilon_t] = 0$. Note that the observational error is not fed back into the system (1)–(3).

Imagine an agent who has existed under the conditions of the reference situation discussed above for t^* periods during which the data $\{\tilde{p}_t\}_0^{t^*}$ have been recorded. Beginning in period t^* the econometrician does not only update her estimates of the parameters of the perceived law of motion (1), she also carries out a standard *specification test.* Upon arrival of a new observation $\tilde{p}_t$ for $t > t^*$, the individual tests the null hypothesis that the price process follows an AR(1) against an unspecified alternative:

$$\begin{aligned} H_0 &: \quad p_t \text{ follows AR(1)} \\ H_1 &: \quad p_t \text{ does not follow AR(1).} \end{aligned}$$

A realization of the test statistic which is unlikely to occur given her AR(1) beliefs, then provides an impetus to reconsider her perception of the law of motion. If such an event occurs, the CEE is no longer existing. Under the current design, a RESET test Ramsey is performed to generate evidence against the null hypothesis. That is, the econometrician uses a widely known, well

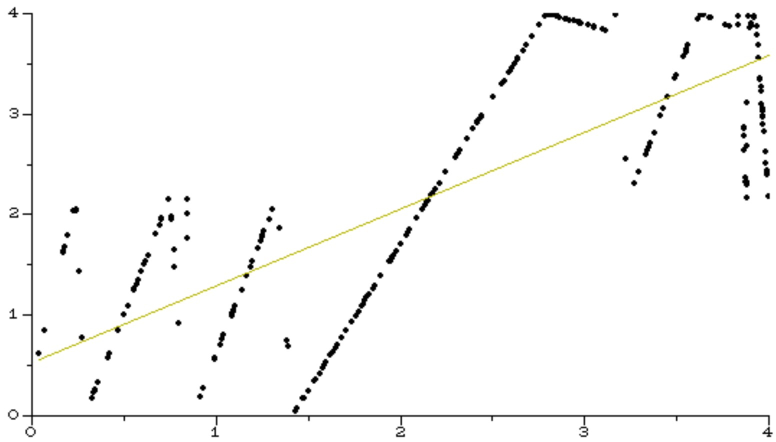

Figure 5: p_t vs. p_{t-4} in the CEE

established tool of time series econometrics. Under this testing strategy, the existence of structure in the residuals from the ordinary least squares fit of the AR(1) is interpreted as evidence against the perceived AR(1) law. Details concerning the test itself as well as the implementation used in our simulations are given in the Appendix. The account given there substantiates the point that the adequate application of the test only requires an understanding of linear statistical concepts.

Let $f_{RESET}(t)$ denote the realization of the test statistic $F_{RESET}(t)$ associated with the RESET test performed at time $t \in (t^*, T)$. This statistic is based on t observed prices. Moreover, consider $\phi(t)$ such that $P(F_{RESET}(t) > \phi_\alpha(t)) = \alpha$ if the null hypothesis is true. Typically α is chosen to be small. The outcome of a single simulation is then summarized in a plot of $f_{RESET}(t)$ against t superimposed on a plot of $\phi_\alpha(t)$ versus t for all $t \in (t^*, T)$. If at a given time t it is observed that $f_{RESET}(t) > \phi_\alpha(t)$, then according to the decision rule of the RESET test, the agent will reject the null hypothesis, since the observed test statistic constitutes a somewhat rare event in the light of the null hypothesis or the *perceived law of motion*. The agent fails to reject H_0, i.e. continues to believe in the *perceived law* of motion, if $f_{RESET}(t) \leq \phi_\alpha(t)$.

The graph exhibited in Figure 9 reflects the outcome of an experiment for which the error distribution was specified as $\epsilon_t \sim u[a, b] = u[-0.2107, 0.2107]$. This choice implies a noise-to-signal ratio (NSR) of $\frac{\hat{\sigma}^2_\epsilon}{\hat{\sigma}^2_{cee}} \approx 0.01$. The probability for rejecting H_0 although it is true is controlled at 5% ($\alpha = 0.05$). In addition, a "burn-in" phase of 10 periods ($t^* = 10$) is defined. The simulation process is terminated after $T = 250$ prices have been observed. Examining Figure 9, we find that $f_{RESET}(t) > \phi_\alpha(t)$ for all $t \in (t^*, T)$. The econometrician using

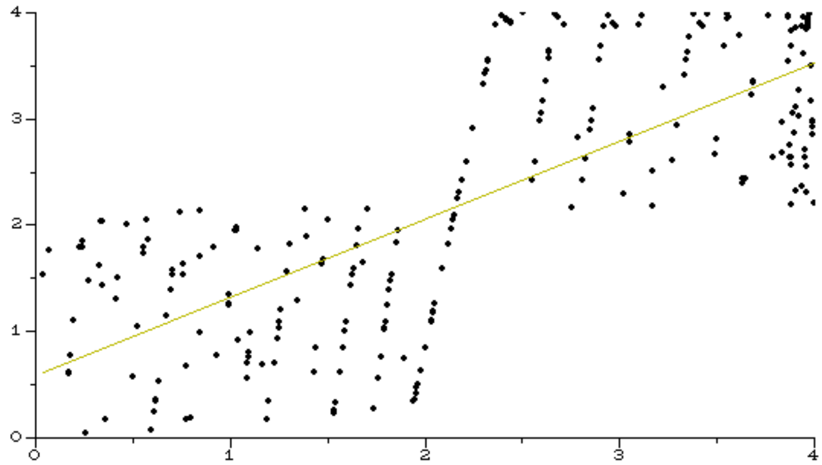

Figure 6: p_t vs. p_{t-8} in the CEE

the RESET test in a chaotic CEE scenario rejects the null hypothesis at each and every point in time. As more and more price data become available, the strength of the evidence against the null hypothesis tends to increase. Note that there exist observations whose consideration as evidence leads to a decrease of the RESET test statistic.

The evidence against the AR(1) becomes less clear once we increase the noise to signal ratio by extending the support of $u[-a, a]$ by considering $a = .4712$ (NSR $=$ .05), $a = 0.6664$ (NSR $=$ .10) and $a = 1.4901$ (NSR $=$ .50). The results for one simulation run for each case are shown in Figures 10–12. In the extreme case of a noise-to-signal ratio of 0.5, the test does not signalize the need to reconsider the perceived law of motion of the price process.

Each of the graphs shown in Figures 10–13 is just one possible outcome of the respective experiment. Replication of the experiment at each design point allows us to obtain estimates of certain features of the process. We are interested in the time elapsed until the test indicates, for the first time, the presence of evidence against the AR(1) belief. Let us define T_R as that point in time at which the first rejection occurs. The set $\{1, 2, 3, \ldots, 250\}$ contains all possible realizations of this discrete random variable. For a given NSR scenario, the simulation experiment is replicated 500 times. The density $f_{T_R}(t_R)$ is then estimated on the basis of these 500 realizations of T_R. The resulting histograms $\hat{f}(t_R)$ are depicted in Figures 13–16.

In the case of the low noise scenario (NSR $= 0.01$), the estimated probability for the event that the econometrician will reject the null hypothesis already at

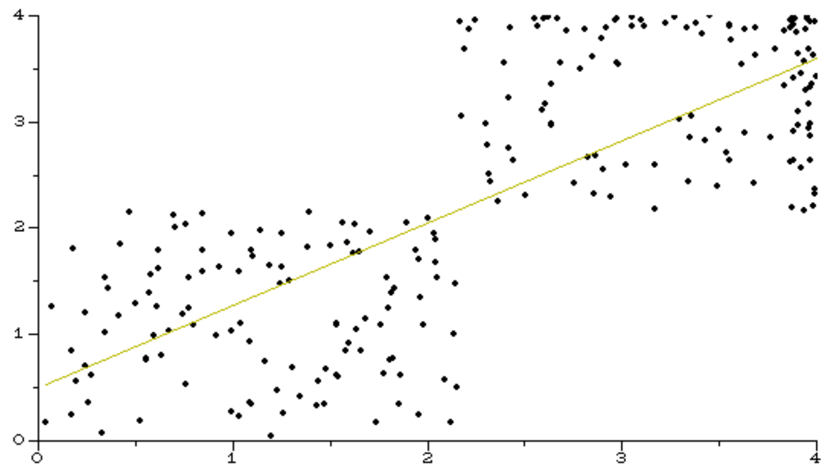

Figure 7: p_t vs. p_{t-20} in the CEE

the first instance of carrying out the RESET test equals 0.94. The relative frequency of the event $t_R > 5$ equals 0.

The histogram for the 5% noise-to-signal ratio, exhibited in Figure 14, indicates that there are few cases in which no evidence against the perceived law was obtained throughout 30 periods. But according to the histogram, the bulk of the probability mass is concentrated over the interval $[1, 10]$. The density estimates for the cases of noise-to-signal ratios of 10% and 50% are given in Figures 15 and 16. Under an increase of the noise-to-signal ratio, the mass gets more evenly distributed over the entire time period $[1, 250]$. Although, one should note that even in the extreme noise-to-signal ratio scenario, rejections may occur at small sample sizes.

In Table 2, I present the estimates for the mean and the variance of the time of first rejection t_R.

Given the experimental evidence, we conclude that the average time until an econometrician operating under the conditions of the CEE example discussed in Hommes/Sorger (1998) is confronted with evidence against his perception of the price process decreases in the quality of the data. For virtually "clean" data, there will be overwhelming evidence against AR(1) beliefs once a standard specification test deeply rooted in the linear paradigm is applied.

By including results from two experiments, we have tried to robustify our argument. The insight that the chaotic CEE is a temporary phenomenon does not rely on an assumption concerning a statistician using a specific linear statistical technology. While the second experiment is much more in line with the sequential character of the original model and perhaps reflects very well

τ	ρ_τ	$\hat{\rho}_\tau$	$\hat{\sigma}_{\hat{\gamma}_1}$
1	- 0.9400	-0.9219	0.0249
2	0.8836	0.8496	0.0338
3	- 0.8306	-0.8078	0.0376
4	0.7807	0.7642	0.0413
5	- 0.7339	-0.7557	0.0419
6	0.6899	0.7508	0.0420
7	- 0.6485	-0.7484	0.0419
8	0.6096	0.7359	0.0429
9	- 0.5730	-0.7345	0.0430
10	0.5386	0.7365	0.0429

Table 1: ρ_τ and sample autocorrelation coefficients $\hat{\rho}_\tau$ based on $\{p_t\}_0^{250}$ generated in CEE

$\frac{\hat{\sigma}_e^2}{\hat{\sigma}_{cee}^2}$	a	b	$\hat{E}[T_R]$	$\hat{V}[T_R]$
.01	-0.2107	0.2107	1.115	0.2128
.05	-0.4712	0.4712	3.755	25.7738
.10	-0.6664	0.6664	7.350	102.1460
.50	-1.4901	1.4901	90.070	8081.8443

Table 2: Estimated $E[T_R]$ and $V[T_R]$ for RESET test in CEE. Note: The variance of the CEE series is approximated at $\hat{\sigma}^2_{CEE} = 1.4803$.

how standard statistical tests are applied in practice—all aspects of sequential testing are, for instance, ignored—the first experiment provides the basis for illustrating a crucial point in our argument. The observation that in the chaotic CEE of the Hommes and Sorger type, agents have learned to believe in a low order autoregressive model, is due to the fact that some properties of that model coincide with certain linear properties of the true underlying dynamical system. Our plots of lagged prices (Figures 2–7) back the conjecture that even a low-tech statistical approach would readily reveal that maintaining AR(1) beliefs is problematic. To model an individual with the ability to act according to a complicated updating scheme but to deny her the ability to grasp that the linear fits presented in Figures 2–7 are poor representations of the underlying dynamics, appears to be problematic.

4. Conclusion and Outlook

In connection with complicated (chaotic) consistent expectations equilibria, it has been claimed that agents using linear statistical techniques are not able to discriminate between the *actual law of motion* and their *perceived law of motion*, a low-order linear autoregression. On the basis of a price trajectory, generated in a complex consistent expectations equilibrium, I demonstrate that an individual

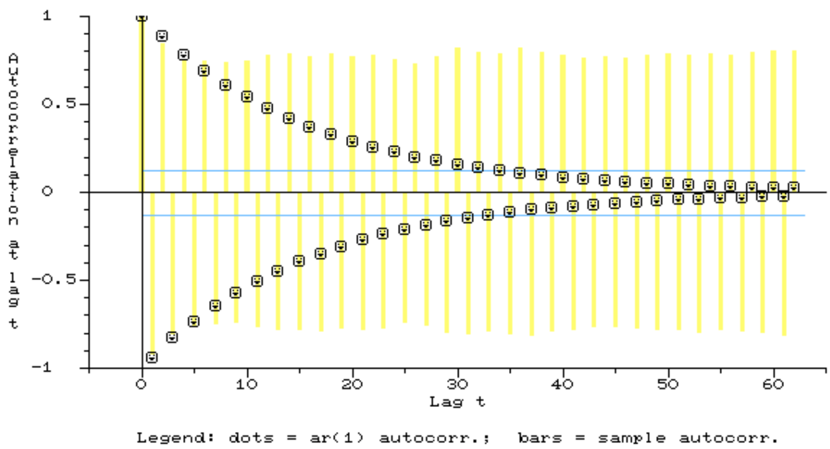

Figure 8: Autocorrelation function versus correlogram

who is equipped with the most basic concepts of statistical inference and some computational equipment will be able to *learn* that her initial perception of the law of motion is not matched by the observed price data. It is shown that a chaotic CEE is a temporary phenomenon once agents are modeled as behaving like econometricians whose competence is restricted to the realm of the linear statistical paradigm. In that sense, the experiments reveal a *lack of robustness* of the chaotic CEE concept. The most interesting property of this concept—despite the fact that they have access to data generated by the *actual law* of motion, agents continue to believe in a *perceived law* which differs from the actual one in a fundamental way—vanishes, once the econometricians engage in a broader spectrum of inferential activities.

Two implications become evident. First of all, if one has the intention of modeling the evolution of forecasting rules in the spirit of Hommes and Sorger, then one should not employ the imagery of a *linear statistician.* On the other hand, if we are interested in studying the dynamics of self-referential systems in which learning behavior plays a role, then we should introduce *sequential* formulations of *specification tests* into the model, thereby allowing for the *accumulation of evidence* against a *perceived law of motion.*

Appendix: Ramsey's Reset Test

Ramsey's RESET test is perhaps one of the most frequently applied specification tests. Based on earlier treatments of model diagnostics in linear models by Anscombe (1961) and Anscombe/Tuckey (1963), Ramsey (1969) devised a

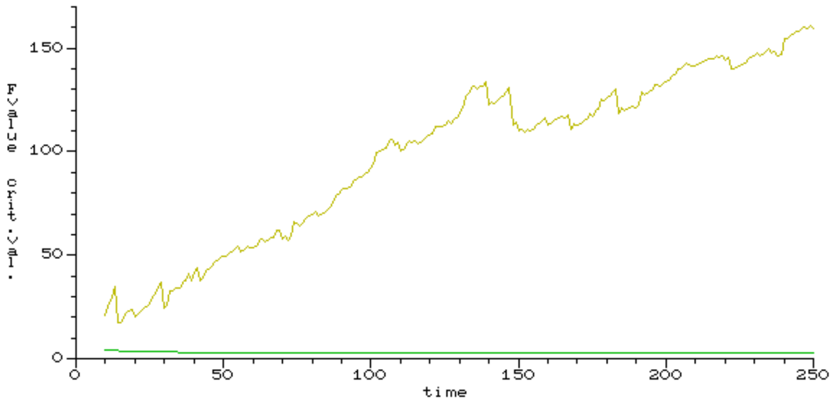

Figure 9: $f_{RESET}(t)$, $\phi_\alpha(t)$ versus t (NSR = .01)

test for functional form. A series of papers (e.g. Ramsey/Gilbert 1969; Ramsey/ Schmidt 1976) devised a statistical test, which basically checks whether or not there is structure in the residuals resulting from the fit of a linear model. The procedure has been repeatedly refined and its properties have been established. Operational characteristics of the RESET test in the presence of nonlinear economic data generating processes are studied in Jungeilges (1998, Chapter 3).

Virtually all of the early versions of the RESET test relied on the assumption of independently, identically, and *normally* distributed errors. But it can be established that Ramsey's test is indeed adequate in a more general setting. It will be demonstrated how the RESET test can be used to test the hypothesis that a given time series is generated by a linear (autoregressive) process. After presenting the AR(p) processes as a linear model of the type encountered in the previous sections, we assume i.i.d. residuals and apply results based on large sample theory to demonstrate that the strategy for the construction of a test for nonlinearity outlined above is implementable in this more general situation. We provide the RESET test statistic and its asymptotic null distribution.

Consider the following augmented AR(l) model:

$$y_t = \sum_{s=1}^{l} \beta_s y_{t-s} + \sum_{j=1}^{p} \phi c_{tj} + \epsilon_t \tag{8}$$

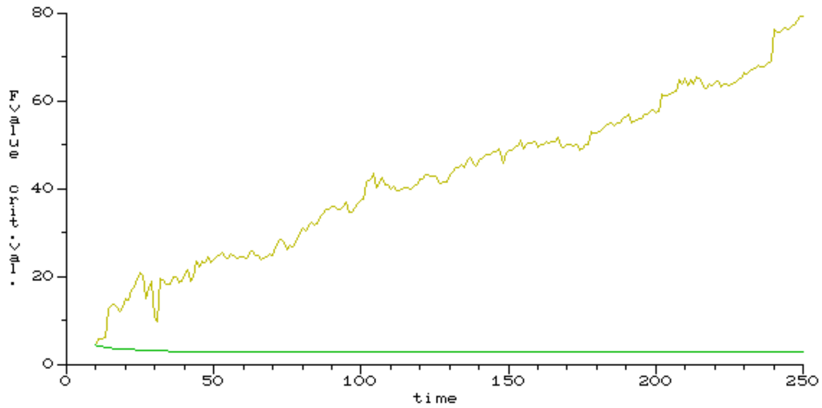

Figure 10: $f_{RESET}(t)$, $\phi_\alpha(t)$ versus t (NSR = .05)

$$\begin{pmatrix} y_t \\ y_{t+1} \\ y_{t+2} \\ \vdots \\ y_T \end{pmatrix} = \left(\begin{array}{cccc|ccc} y_{t-1} & y_{t-2} & \ldots & y_{t-l} & c_{t1} & \ldots & c_{tp} \\ y_t & y_{t-1} & \ldots & y_{t-(l+1)} & c_{t+1,1} & \ldots & c_{t+1,p} \\ y_{t+1} & y_t & \ldots & y_{t-(l+2)} & c_{t+2,1} & \ldots & c_{t+2,p} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ y_T & y_{T-1} & \ldots & y_{T-l} & c_{T,1} & \ldots & c_{T,p} \end{array}\right) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_l \\ \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ \epsilon_{t+1} \\ \epsilon_{t+2} \\ \vdots \\ \epsilon_T \end{pmatrix} \tag{9}$$

$$Y_{(T-l)\times 1} = (X_{(T-l),l} | \Phi_{(T-l),p}) \begin{pmatrix} \beta \\ \phi \end{pmatrix} + e \tag{10}$$

With respect to the error structure as well as the error distribution, the following is assumed to hold:

Assumption 1 (Moment conditions) *Let e denote a $(T, 1)$ vector of random variables ϵ_t, $t = 1, 2, \ldots, T$. Then*

1. $E[e] = 0$,
2. $V[e] = \sigma^2 I_{T,T}$ *with* $\sigma^2 < \infty$.
3. $E[\epsilon_t^4]$ *exists for all* $t = 1, 2, \ldots$
4. $E[\epsilon_t^4] < m$ *where* m *denotes a finite real constant.*

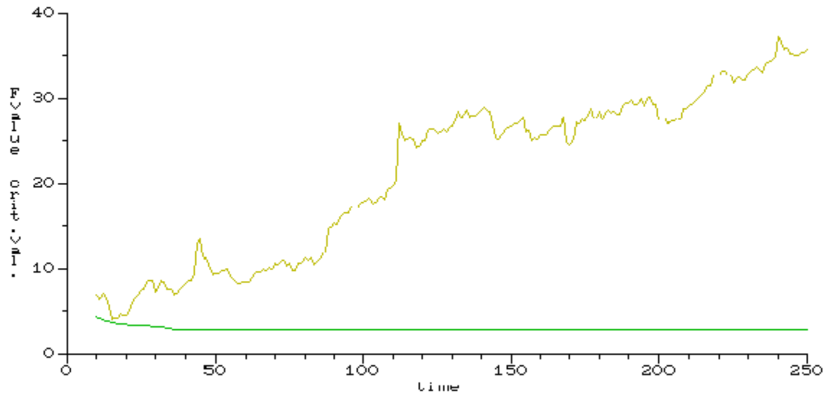

Figure 11: $f_{RESET}(t)$, $\phi_\alpha(t)$ versus t (NSR = .10)

The usual stationarity assumption holds for the linear difference part of the model.

Assumption 2 (Stationarity condition) *The roots of the polynomial*

$$a(z) = 1 - \sum_{j=1}^{l} \beta_j z^j$$

lie outside the unit circle $\mid z \mid = 1$.

Given the validity of assumptions (1) and (2) it can be shown that

$$\sqrt{T}\begin{pmatrix} \hat{\beta} - \beta \\ \hat{\phi} - \phi \end{pmatrix} \sim \mathcal{MVN}(0, \sigma^2 V^{-1}) \tag{11}$$

where $\mathcal{MVN}(\bullet, \bullet)$ denotes the family of multivariate normal distributions and V

$$V = plim_{T\to\infty} \frac{1}{T}\begin{pmatrix} X_T' X_T & X_T' \Phi_T \\ \Phi_T' X_T & \Phi_T' \Phi_T \end{pmatrix}. \tag{12}$$

The variance-covariance matrix is approximated by

$$s^2 V^* = s^2 \Big[\frac{1}{T}\begin{pmatrix} X_T' X_T & X_T' \Phi_T \\ \Phi_T' X_T & \Phi_T' \Phi_T \end{pmatrix}\Big]^{-1} \tag{13}$$

Then for large T

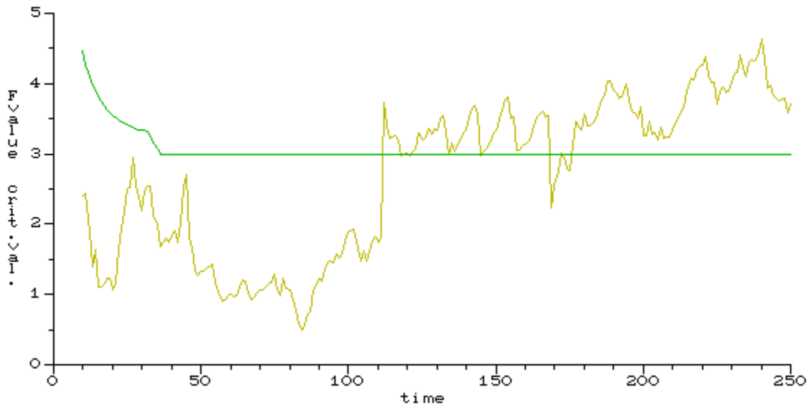

Figure 12: $f_{RESET}(t)$, $\phi_\alpha(t)$ versus t (NSR = .50)

$$\begin{pmatrix} \hat{\beta} \\ \hat{\phi} \end{pmatrix} \sim \mathcal{MVN}(\begin{pmatrix} \beta \\ \phi \end{pmatrix}, \frac{\sigma^2}{T}(V^*)^{-1}). \tag{14}$$

For details see Anderson (1971, Sec. 5.5).

Consequently, we can use results established in connection with tests of the general linear hypothesis to facilitate the derivation of the null distribution for the test. Ramsey's procedure amounts to testing

$$H_0 : (0_{(p\times l)}|I_{p\times p}) \begin{pmatrix} \beta \\ \phi \end{pmatrix} = 0 \tag{15}$$

$$H_1 : (0_{(p\times l)}|I_{p\times p}) \begin{pmatrix} \beta \\ \phi \end{pmatrix} \neq 0. \tag{16}$$

A test for a nonlinear generating mechanism which is consistent with the approach propagated by Ramsey and Schmidt then relies on a test statistic and its distributional properties under H_0, which is given in the result stated below.

Result 1 *Consider the augmented autoregressive model (8). Let assumptions (1) and (2) hold. As* $T \to \infty$

$$RESET_T = (T - l - r[X|\Phi])\frac{SSE_K - SSE}{SSE} \xrightarrow{\mathcal{P}} \chi^2(p) \tag{17}$$

Considering the relationship between the χ^2 - distribution and the F - distribution, it follows that asymptotically $RESET_T$ is equivalent to the standard Ramsey statistic F_{RESET} (see Algorithm $RESET$). In the experiments, I

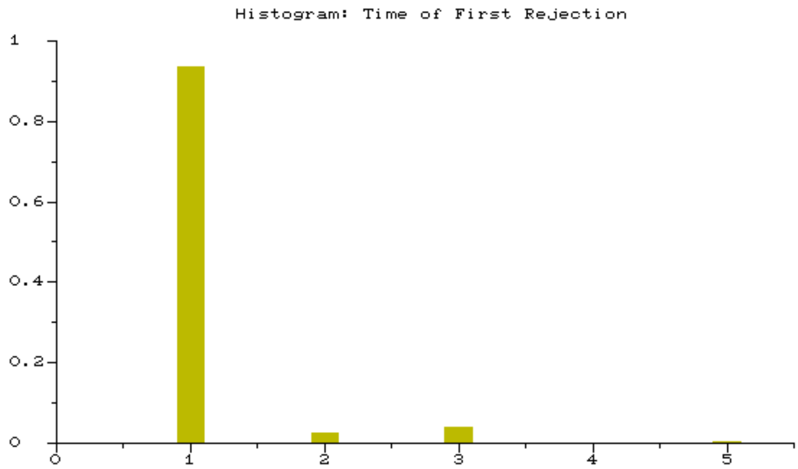

Figure 13: Relative Frequency Histogram for t_R (NSR $= .01$)

did not only use the percentage point of this asymptotic null as critical value. For small sample sizes, $\phi(t)$ is based on respective percentage points derived in Jungeilges Jungeilges (1998, Chapter 3).
The algorithm listed below describes the implementation of the test used in the simulation experiment. The augmentation matrix Φ is constituted by powers of $\hat{Y}$, the one-step-ahead forecasts on Y.

ALGORITHM: RESET

Functions	$Y\ OLSRES\ X$: generates OLS estimates $\hat{\beta}$ and e for linear model $Y = X\beta + \epsilon$
	$ORDERSEL\ Y$: given time series Y, the order of an $AR(l)$ and the associated regressor matrix X is determined
Input	$Y \leftarrow data \ \diamond\ p \leftarrow$ number of powers to be included
Step 1	$l \leftarrow ORDERSEL\ Y$
Step 2	$e \leftarrow Y\ \ OLSRES\ X\ \ \diamond SSEK \leftarrow e'e\ \diamond\ \hat{Y} \leftarrow X\hat{\theta}$
Step 3	$\Phi \leftarrow (\hat{Y}^2,\ \hat{Y}^3,\ \ldots,\ \hat{Y}^{p+1})$
Step 4	$u \leftarrow Y\ OLSRES\ (X \mid \Phi)\ \diamond\ SSE \leftarrow u'u$
Step 5	$F_{RESET} \leftarrow \frac{N-r(X\mid\Phi)}{p}\frac{SSK-SSE}{SSE}$

In the simulations, it is assumed that the order of the autoregressive process is known ($l \leftarrow 1$). The order selection function ORDERSEL is overruled. The number of artificial regressors to be included is set equal to 2 ($p \leftarrow 2$).

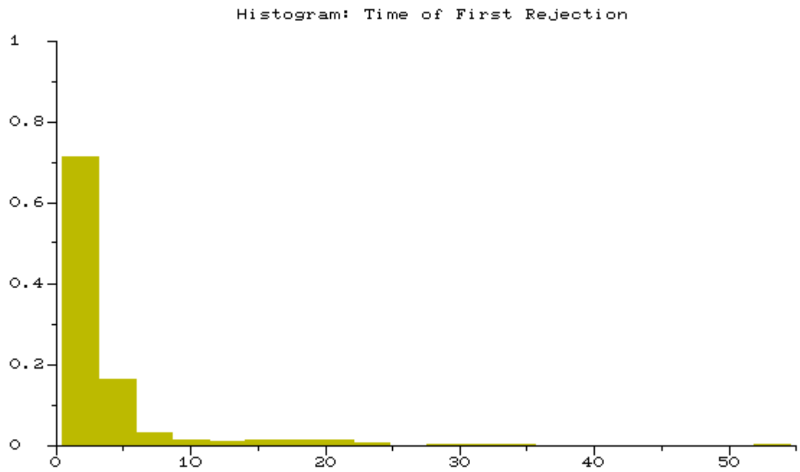

Figure 14: Relative Frequency Histogram for t_R (NSR = .05)

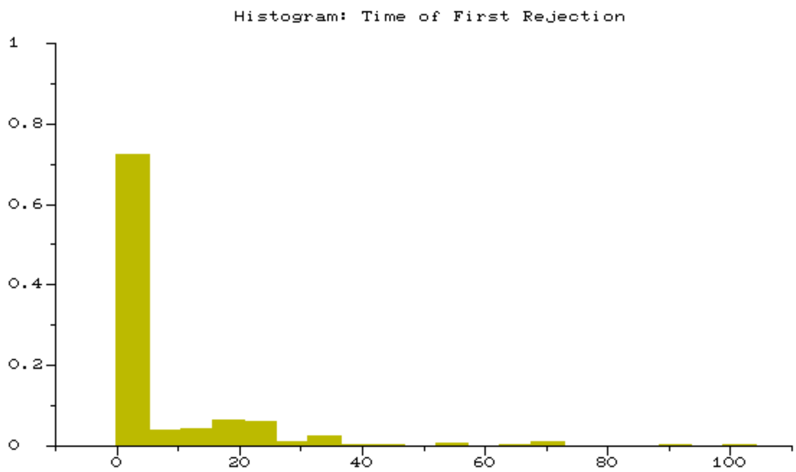

Figure 15: Relative Frequency Histogram for t_R (NSR = .10)

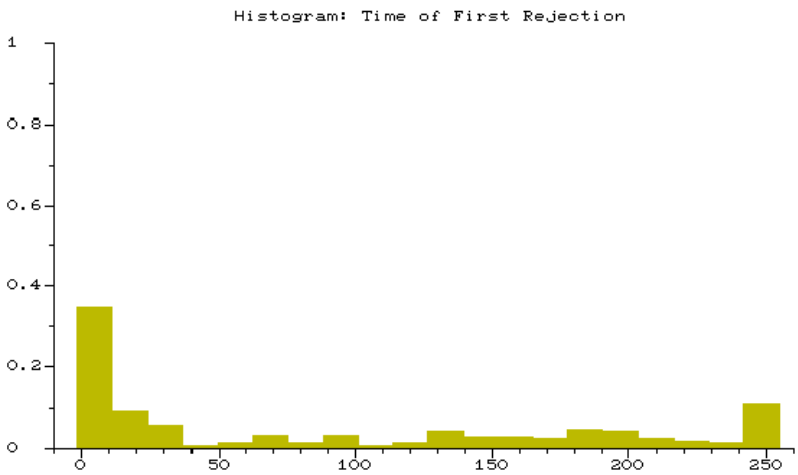

Figure 16: Relative Frequency Histogram for t_R (NSR = .50)

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