Analyse & Kritik 29/2007 (© Lucius & Lucius, Stuttgart) p. 259–268

Eliora van der Hout/Jack Stecher/Harrie de Swart

A Critical Discussion of the Characteristic Properties of List PR and FPTP Systems

Abstract: This paper discusses the characteristic properties of List PR systems and FPTP systems, as given in Hout 2005 and Hout et al. 2006. While many of the properties we consider are common to both systems, it turns out (see Hout 2005) that the British system distinguishes itself by satisfying the *district cancellation* property, while the Dutch system distinguishes itself by satisfying *consistency* and *anonymity*. For scoring rules, topsonlyness is equivalent to being party fragmentation-proof (see Hout 2005; Hout et al. 2006). One might present this as an argument in favour of requiring topsonlyness. However, we will also give counter-arguments against insisting upon the property of being party fragmentation-proof.

0. Introduction

Let V be a finite set of voters; $i, j, \ldots$ denote elements of V. I denotes any non-empty subset of V.

Let A be a finite set of alternatives (parties); $x, y, \ldots$ denote elements of A. $L(A)$ denotes the set of linear orderings, and $W(A)$ the set of weak orderings on A.

A function $c : I \to L(A)$ is called a *profile* on I; $c_i := c(i)$. We write $t(c_i) = x$ to denote that x is at the top of c_i.

Let $\Delta = \{\delta_1, \ldots, \delta_m\}$ be a partition of the set V of voters into constituencies. For each $I \subseteq V$, Δ induces a partition Δ^I of I; $\Delta^I = \{\delta_1 \cap I, \ldots, \delta_m \cap I\}$.

For $c \in L(A)^I$ and $\delta \in \Delta$, c_δ is the restriction of c to $\delta \cap I$.
For c a profile on I, we denote by $\pi_c(x)$ the number of voters in I who have x as first choice in their linear ordering c_i of the alternatives.

Definition 1 For $c : I \to L(A)$, $\pi_c(x) := \mid \{i \in I \mid t(c_i) = x\} \mid$. In particular, $\pi_{c_\delta}(x) = \mid \{i \in \delta \cap I \mid t(c_i) = x\} \mid$.

For every subset I of voters, a social preference rule F assigns a *weak* ordering $F(c)$ to every profile c on I. We state this formally as follows:

Definition 2 A *social preference rule* is a function $F : \bigcup_{I \subseteq V} L(A)^I \to W(A)$. So, for every $I \subseteq V$, $F : L(A)^I \to W(A)$. (It is understood that F does not depend on the particular set I of voters.)

Definition 3 By $t(F(c))$ we mean the set of top elements of the weak ordering $F(c)$.

$$t(F(c)) := \{x \in A \mid x \text{ is a top element of } F(c)\}$$

Next we define the Dutch and British social preference (or aggregation) rules, called D and B respectively.

Definition 4 *:* The *plurality ranking (PR)* rule is the function $D : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{D(c)} y$ iff $\pi_c(x) \geq \pi_c(y)$ (D for Dutch).

Definition 5 Alternative x is said to *win* in district δ given profile c if x is ranked first in δ by the largest number of voters; i.e., if for all $z \in A$, $\pi_{c_\delta}(x) \geq \pi_{c_\delta}(z)$.

Definition 6 $\tau_c^\Delta(x)$ denotes the number of constituencies in which x wins given profile c.

$$\tau_c^\Delta(x) := \mid \{\delta \cap I \in \Delta^I \mid x \text{ wins in } \delta \text{ given } c\} \mid .$$

Definition 7 Given Δ, the *First-Past-The-Post (FPTP)* system is the function $B : L(A)^I \to W(A)$, for $I \subseteq V$, defined by $x \succeq_{B(c)} y$ iff $\tau_c^\Delta(x) \geq \tau_c^\Delta(y)$ (B for British).

1. Characterizations

Theorems 1 and 2 below characterize the Dutch and British aggregation rules, respectively. The properties in question will subsequently be defined.

Theorem 1 Let F be a social preference rule. F is the plurality ranking rule D if and only if F is *consistent*, faithful, *anonymous*, neutral, and topsonly.
In addition, these properties are independent.

For the proof, we refer to Hout 2005; Hout et al. 2006. The properties printed in italics distinguish the plurality ranking rule from the FPTP system, as we will see further on.

Theorem 2 Let F be a social preference rule. F is the FPTP system B if and only if F is subset consistent, district consistent, subset anonymous, neutral, topsonly, Paretian, and satisfies *district cancellation.* In addition, these properties are independent.

For the proof, we refer to Hout 2005; Hout and de Swart 2007. The property printed in italics distinguishes the FPTP system from the plurality ranking rule, as we will see further on. The following propositions are also proved in Hout 2005; Hout and de Swart 2007.

Proposition 1 The PR rule D satisfies all properties of B except *district cancellation.*

Proposition 2 The FPTP system B satisfies all properties of the plurality ranking rule D except *FS cancellation,* consistency and *anonymity.*

We initially focus on the cancellation properties.

Definition 8 F satisfies *district cancellation* iff for all $I \subseteq V$, for all profiles c on I and for all $x, y \in A$, if $|\{\delta \in \Delta | x \in t(F(c_\delta))\}| = |\{\delta \in \Delta | y \in t(F(c_\delta))\}|$, then $x \sim_{F(c)} y$.

Lemma 1 Clearly, B satisfies district cancellation, but D does not.

Counterexample 1: $A = \{x, y, z\}$, $\Delta = \{\delta\}$, $\delta = \{1, 2, 3\}$, 1: $x \ldots$; 2: $x \ldots$; and 3: $y \ldots$. Then $t(D(c_\delta)) = \{x\}$. So, the number of districts with y at the top of $D(c_\delta)$ is 0. The same holds for z. But not $y \sim_{D(c)} z$: y has more first votes than z.
However, D does satisfy FS cancellation.

Definition 9 A social preference rule F has the *FS cancellation property* if, for all $I \subseteq V$, for every $c \in L(A)^I$, and for all $x, y \in A$, if $\pi_c(x) = \pi_c(y)$, then $x \sim_{F(c)} y$.

Lemma 2 Clearly, D does satisfy FS cancellation, but B does not.

Counterexample 2: Let $A = \{x, y, z\}$, $I = V = \{1, \ldots, 6\}$, $\Delta = \{\delta_1, \delta_2\}$ with $\delta_1 = \{1, 2, 3\}$ and $\delta_2 = \{4, 5, 6\}$. Suppose $t(c_1) = t(c_2) = x$, $t(c_3) = t(c_4) = y$ and $t(c_5) = t(c_6) = z$. Then $\pi_c(x) = \pi_c(y) = 2$, but not $x \sim_{B(c)} y$, since x wins in district δ_1, while y wins in no district.

Topsonlyness requires that, whenever the tops of the individual preference orderings match for two profiles, the social preference rule should choose the same outcome for both profiles.

Definition 10 A social preference rule F is *topsonly* if, whenever $c, c' \in L(A)^I$ are such that for all $i \in I$ and for all $x \in A$, $t(c_i) = x$ iff $t(c'_i) = x$, then $F(c) = F(c')$.

In Hout 2005; Hout et al. 2006, the following Lemma and Theorem are shown.

Lemma 3 If F is anonymous, neutral, and topsonly, then F has the FS cancellation property.

Theorem 3 Let $F : L(A)^I \to W(A)$, for $I \subseteq V$, be a social preference rule. F is the plurality ranking rule D iff F is consistent, faithful, and has the FS cancellation property.

Remark 1 *(Argument against topsonlyness)* The FS Cancellation Property and topsonlyness seem somehow unintuitive: requiring these properties amounts to saying that breadth of support counts and depth of opposition does not. For example, suppose that in Germany in 1933, an equal number of people were to support von Schleicher and Hitler as their first choice. Cancellation and topsonlyness would make the society indifferent between the two, but presumably those who objected to Hitler objected much more strongly than those who objected to von Schleicher.

By *consistency* of F we mean that if two disjoint sets of voters I and J both socially prefer x to y using F, their union should also socially prefer x to y using F. More precisely (in order to be able to prove Theorem 1):

Definition 11 A social preference rule F is *consistent* iff, whenever $c \in L(A)^I$, $c' \in L(A)^J$ are preference profiles for disjoint sets of voters I and J and $c + c'$ is the profile on $I \cup J$ that corresponds with c on I and with c' on J, then for all $x, y \in A$: if $x \succeq_{F(c)} y$ and $x \succ_{F(c')} y$, then $x \succ_{F(c+c')} y$.

Lemma 4 Clearly, D is consistent, but B is not.

Counterexample 3: $\delta_1 = \{1\}$, $\delta_2 = \{2,3,4\}$, $I = \{1,2,3\}$, $J = \{4\}$.

	1	δ_1	x		x
I	2		y		y
	3	δ_2	y		y
J	4			x	x
			c	c'	$c+c'$

$\delta_2 \cap I \neq \emptyset$, $\delta_2 \cap J \neq \emptyset$.

$x \sim_{B(c)} y$ and $x \succ_{B(c')} y$, but not $x \succ_{B(c+c')} y$.

However, B satisfies subset consistency and district consistency, which we now define.

Subset consistency requires that if two sets of voters within the same constituency use the same social preference rule and the tops of the social orderings they choose have at least one element in common, then the top of the social ordering that their union chooses should contain exactly the shared elements.

Definition 12 Let $I^\delta \subseteq \delta \subseteq V$ and $J^\delta \subseteq \delta \subseteq V$ be disjoint subsets of one single constituency δ. A social preference rule is *subset consistent* if, whenever $c_\delta \in L(A)^{I^\delta}$ and $c'_\delta \in L(A)^{J^\delta}$ are preference profiles on I^δ and J^δ, and $c_\delta + c'_\delta$ is the profile on $I^\delta \cup J^\delta$ that corresponds to c_δ on I^δ and to c'_δ on J^δ, $t(F(c_\delta)) \cap t(F(c'_\delta)) \neq \emptyset$ implies $t(F(c_\delta + c'_\delta)) = t(F(c_\delta)) \cap t(F(c'_\delta))$.

District consistency is similar to consistency as defined above in Definition 11, with the proviso that the two sets of voters in question are not only disjoint, but also do not both contain elements of a common constituency. Note that this latter condition is not satisfied in counterexample 3 above, where I and J both contain elements of the same constituency δ_2.

Definition 13 Let $c \in L(A)^I$ and $c' \in L(A)^J$ be preference profiles on disjoint sets of voters $I, J \subseteq V$. Suppose that, for all $\delta \in \Delta$, $\delta \cap I \neq \emptyset$ implies $\delta \cap J = \emptyset$. Let $c + c'$ be the profile on $I \cup J$ that corresponds with c on I and c' on J. Then a social preference rule F is *district consistent* if, for all $x, y \in A$, if $x \succ_{F(c)} y$ and $x \succeq_{F(c')} y$, then $x \succ_{F(c+c')} y$.

Anonymity means that it does not matter who casts which vote; the names of the voters are irrelevant. In other words, all voters are treated equally.

Definition 14 F is *anonymous* := for all $I \subseteq V$, for every permutation σ of I, and for all preference profiles $c \in L(A)^I$, $F(c \circ \sigma) = F(c)$.

A social preference rule is called *subset anonymous* if it treats all voters equally when society consists of (a subset of) one single constituency.

Definition 15 A social preference rule F is *subset anonymous* if, for all $\delta \in \Delta$, for all $I^\delta \subseteq \delta$, for every permutation σ of I^δ, and for all preference profiles $c_\delta \in L(A)^{I^\delta}$, $F(c_\delta \circ \sigma) = F(c_\delta)$.

Lemma 5 Clearly, D is anonymous, but B is not.

Counterexample 4: Let $A = \{x, y, z\}$, $I = V = \{1, \ldots, 6\}$, $\Delta = \{\delta_1, \delta_2\}$ with $\delta_1 = \{1, 2, 3\}$ and $\delta_2 = \{4, 5, 6\}$, $\sigma(1) = 4$ and $\sigma(4) = 1$. Let c be as indicated in the table below.

		c	$c \circ \sigma$
	1	$x \ldots$	$y \ldots$
δ_1	2	$x \ldots$	$x \ldots$
	3	$y \ldots$	$y \ldots$
	4	$y \ldots$	$x \ldots$
δ_2	5	$z \ldots$	$z \ldots$
	6	$z \ldots$	$z \ldots$

$\tau_c^\Delta(x) = 1$ and $\tau_c^\Delta(y) = 0$ and $\tau_c^\Delta(z) = 1$. But $\tau_{c\circ\sigma}^\Delta(x) = 0$ and $\tau_{c\circ\sigma}^\Delta(y) = 1$ and $\tau_{c\circ\sigma}^\Delta(z) = 1$. Hence, $B(c \circ \sigma) \neq B(c)$.
So, B is not anonymous, but it is clearly subset anonymous.

Summarizing:

- D and B have the following properties in common: faithful, neutral, tops-only, subset consistent, district consistent, subset anonymous, Pareto optimal.

- D is consistent and anonymous (and satisfies FS cancellation), but does not satisfy district cancellation.

- B satisfies district cancellation (but not FS cancellation), but is not consistent or anonymous.

consistency:	D	counterexample 3
subset consistency:	D	B
district consistency:	D	B
FS cancellation:	D	counterexample 2
district cancellation:	counterexample 1	B
anonymous:	D	counterexample 4
subset anonymous:	D	B

In order to make this paper self-contained, below we will give precise definitions of the properties mentioned which have not been treated earlier in this paper.

A social preference rule F is *faithful* if, in case society consists of a single individual whose most preferred party is x, it orders this party x first. More precisely:

Definition 16 A social preference rule F is *faithful* iff, for all $i \in V$, for all $c_i \in L(A)^{\{i\}}$, and for all $x \in A$, if $t(c_i) = x$, then $t(F(c_i)) = x$.

Neutrality means that all parties are treated equally.

Definition 17 A social preference rule F is *neutral* if, for every permutation λ of A, for all $I \subseteq V$, and for every preference profile $c \in L(A)^I$, $F(\lambda c) = \lambda F(c)$.

Pareto optimality requires that, whenever all individuals prefer x to y, then y is not ranked first socially.

Definition 18 A social preference rule F is *Pareto optimal* if, for all parties $x, y \in A$, for all $I \subseteq V$, and for all preference profiles $c \in L(A)^I$: if $x \succ_{c_i} y$ for all $i \in I$, then $y \notin t(F(c))$.

In Hout 2005; Hout and de Swart 2007, the following Lemma is shown.

Lemma 6 If F is subset anonymous, neutral, topsonly, and Pareto optimal, then F has the subset cancellation property.

As a consequence, both D and B have the subset cancellation property. *Subset cancellation* demands that when a set of voters is a subset of one single constituency and all alternatives that receive a nonzero vote total tie, the social preference rule should rank this whole set of alternatives first.

Definition 19 A social preference rule F has the *subset cancellation* property if, for all $\delta \in \Delta$, for all $I^\delta \subseteq \delta \subseteq V$, and for every $c_\delta \in L(A)^{I^\delta}$, if for all parties $x, y \in A$ with $\pi_{c_\delta}(x) \neq 0$ and $\pi_{c_\delta}(y) \neq 0$, $\pi_{c_\delta}(x) = \pi_{c_\delta}(y)$, then $t(F(c_\delta)) = \{x \in A \mid \pi_{c_\delta}(x) \neq 0\}$.

Summarizing:

- D satisfies FS and subset cancellation, but not district cancellation.
- B satisfies district and subset cancellation, but not FS cancellation.

2. Seat Share Allocation

In Remark 1 we have argued against the property of topsonlyness (and of FS cancellation). However, a tentative argument in favour of topsonlyness might be the following.

In representative democracies we can distinguish two stages of decision-making : (1) the electorate chooses representatives; (2) the representatives make binding decisions. Instead of modelling List PR systems by the plurality ranking rule, we might model them by a seat share allocation rule, where a *seat share allocation rule* is a function that assigns to each profile for each party a seat share for this party. One might argue that such a seat share allocation rule should be *party fragmentation-proof;* that is, a party x cannot obtain a larger seat share by splitting up into two parties x_1 and x_2 with similar policy positions.

The Borda rule has all characteristic properties of D except topsonlyness, but it is not party fragmentation-proof, as becomes clear from the following example:

Example Hout and McGann 2004: The Borda rule is not party fragmentation-proof. Suppose $A = \{x, y\}$ and the four voters $1, \ldots, 4$ vote as follows.

Voter	1	2	3	4
	x	x	y	y
	y	y	x	x

Then Borda score $(x) = 2$ and Borda score $(y) = 2$. So, x and y would receive an equal allocation of seats. Now suppose x splits into x_1 and x_2 with similar policy positions. Then these 4 voters will (probably) have the following preference orderings.

Voter	1	2	3	4
	x_1	x_2	y	y
	x_2	x_1	x_2	x_1
	y	y	x_1	x_2

But then Borda score $(x_1) = 4$, Borda score $(x_2) = 4$, and Borda score $(y) = 4$. So, x_1 and x_2 together would receive twice the seat share of y.

In Hout 2005; Hout and de Swart 2007, the following theorem is proved.

Theorem 4 A scoring seat share allocation rule is party fragmentation-proof if and only if it is topsonly.

One might be tempted to see this theorem as an argument in favour of topsonlyness and against the Borda rule. On the other hand, we present below an argument against requiring this property of being party fragmentation-proof.

Remark 2 (*Arguments against party fragmentation-proofness*) It seems reasonable to expect, under a non-topsonly seat allocation rule, that parties will fragment until the gain from fragmentation is no longer unambiguously positive. If so, the desirability of being fragmentation-proof is unclear: there would be fragmentation to the point of allowing voters to support their desired wing of a party, without so much fragmentation as to cause complete disarray. Fragmentation may give voters a more precise way of expressing their desires.

In the above example, if party x can increase its share of seats by splitting into x_1 and x_2 at the expense of party y, then party y can do the same thing. If the eventual outcome is to give the voters the largest possible set of alternatives (at least relative to their preferences), is that undesirable? It does not seem to harm the voters, because they can form a complete linear ordering over an arbitrary list of parties.

Additionally, it may be possible that there is an upper bound on a given party's share of seats, no matter how many times it fragments.

Finally, we give below the precise definitions of the notions involved.

Definition 20 Let m be the number of alternatives in A and $v = \langle v_1, \ldots, v_m \rangle$ be a scoring vector (i.e., $v_1 \geq v_2 \geq \ldots \geq v_m \geq 0$ and $v_1 > 0$). A *scoring seat allocation rule* is a function $F_v : L(A)^I \to [0,1]^A$, defined as follows: $F_v(c) : A \to [0,1]$ is the seat share function that assigns to any party $x \in A$ its seat share $\frac{\tau_{v,c}(x)}{\tau_v(c)}$. Here $\tau_{v,c_i}(x) = v_k$ iff x is the kth preference of voter i. $\tau_{v,c}(x)$ is the sum of the scores for party $x \in A$ over all the individuals in profile c, and $\tau_v(c)$ is the total score of a profile c, summed over all parties.

Let $A' = (A - \{x\}) \cup \{x_1, x_2\}$ and let $c' \in L(A')^I$ be the profile that corresponds with profile $c \in L(A)^I$, except that party x_1 and party x_2 take the position of party x in the preference orderings of the voters. So, if for example x is ordered second by some individual at c, then x_1 is ordered second and x_2 is ordered third at c' or x_2 is ordered second and x_1 is ordered third at c'.

Definition 21 A seat allocation rule $F_v : L(A)^I \to [0,1]^A$ is *party fragmentation-proof* if there exist no party x and profile c such that $F_{v'}(c')(x_1) + F_{v'}(c')(x_2) > F_v(c)(x)$, where x_1 and x_2 result from splitting up party x in two parties with similar policy positions, c' results from c as described above and v' is some scoring vector $\langle v_1, v_2, \ldots, v_m, v_{m+1} \rangle$ if $v = \langle v_1, v_2, \ldots, v_m \rangle$.

Definition 22 Let $\langle v_1, \ldots, v_m \rangle$ be the scoring vector v. $F_v : L(A)^I \to [0,1]^A$ is *topsonly* iff for all for all $i \in \{2, \ldots, m\}$, $v_i = 0$.

3. Concluding Remarks

In the above, we have presented arguments both in favour and against the properties that distinguish the Dutch PR electoral system from the British FPTP system. We conclude with an elaboration of these arguments.

The principle behind counterexamples 2 and 4 is essentially that of gerrymandering. In any system with electoral districts, there is always the possibility that the districts can be redrawn; this is what in effect counterexample 2 does. Likewise, voters may change their electoral clout by moving across districts, which is what the voter-swapping in counterexample 4 shows. The Dutch List system does not have these issues arise essentially because it has no districts. Thus, the arguments in this paper highlight the well-known conflict between the principles of one-citizen one-vote and one-state one-vote.

The violations of FS cancellation and anonymity, when viewed in this light, are part of the arguments one might make in favour of the FPTP system. For example, on 1 January 2007, the European Union had 30 million new inhabitants join (22 million from Romania and 8 million from Bulgaria), with per capita GDP of just under \$9.000 on a purchasing power parity basis. At the time, EU member Finland had just over 5 million inhabitants, and a per capita GDP of just over \$31.000. Under a system without FS cancellation or anonymity (i.e., one with political districts), Finland was willing to support Bulgaria and Romania as entrants to the EU. The presence of districts provides some degree of protection to small groups, and may make them willing to trade and interact with larger but poorer groups.

In the discussion above about party fragmentation-proofness, there is an implicit distinction between fragmentation along ideological divisions and non-ideological fragmentation. More specifically, if parties fragment for strategic reasons, the splits may be along philosophical or personality lines, in which case voters are given a more precise way of expressing their desires. On the other hand, if fragmentation does not enable voters to choose among different wings of the pre-fragmentation party or to express opinions on some sort of leadership personality clashes, then the fragmentation would seem to benefit only the party and not necessarily the electorate.

Even if one grants that party fragmentation-proofness is desirable, any analysis of the merits of topsonlyness needs to be made cautiously. For example, suppose that parties x and y are the only contenders. The example from Hout and McGann 2004 only shows what happens if x acts in isolation and if y neither responds nor credibly threatens to respond. Unless y cannot also fragment, the party fragmentation in the example may not be equilibrium.

When y can also fragment, it is unclear that any non-ideological fragmentation can occur. For example, if both parties enjoy equal support and can field equally many candidates, then one equilibrium would be for each party to adopt the following fragmentation strategy: retain the current composition as long as the other party does so, and fragment completely (i.e., into individual candidates) if the other party fragments at all. As long as both x and y are no better off with all parties disbanded than they are with the current two parties, it would be an equilibrium for neither party ever to fragment. In practice, party fragmentation seems rare, though the former USSR was able to increase its seat count in the United Nations by fragmenting into fifteen new states. (The improved seat count, however, is unlikely to have had much to do with the fragmentation.)

Suppose, however, that parties do fragment, and consider the extreme case where all parties keep fragmenting in order to improve their outcomes under a Borda-like scheme. What is the result? The fragmentation can end when the voters eventually see a list of independent candidates, with no political parties. This is the ideal that James Madison argued for in Federalist #10, when writing about the destructiveness of factions. If voters are fully empowered to choose their favourite candidates, and find themselves in a world where politicians do not form factions, then it is not immediately obvious that the public suffers.

It should be pointed out that topsonlyness has other properties that are often viewed as desirable. For example, in Berga 2004 it is shown that strategy-proofness is related to topsonlyness in the case of single-plateaued preferences (the condition is called *plateau-onlyness* there).

Hence we do not come down unambiguously in favour of one system over the other, and must be content to characterize the properties of each system. The Dutch and the British have been aware of each other for some time, and have had many chances to observe one another's electoral systems in practice. Presumably if one system were dominant in some sense over the other, we would have seen voters in the country with the dominated system choose to change their electoral procedures. Nevertheless, the formalization of the properties of both that are presented here enable us to clarify the trade-offs faced when choosing between the two systems.

Bibliography

Dolors, B. (2004), Strategy-proofness and Single-plateaued Preferences, in: *Mathematical Social Sciences 35(2)*, 105–120

van der Hout, E. (2005), *Electing Representatives; A Comparison of List PR and FPTP Systems*, Ph.D. thesis, Tilburg University

—/A. McGann (2004), *Equal Protection Implies Proportional Representation*, Institute for Mathematical Behavioral Sciences Paper 10, University of California at Irvine

—/H. de Swart/A. ter Veer (2006), Characteristic Properties of List Proportional Representation Systems, in: *Social Choice and Welfare 27*, 459–475

—/H. de Swart (2007), *Characteristic Properties of FPTP systems*, Submitted for publication. To be found at `http://www.tilburguniversity.nl/faculties/humanities/dphil/staff/swart/papers.html`