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Non-deteriorating Choice Without Full Transitivity*

Abstract: Although the theory of greatest-element rationalizability and maximal-element rationalizability on general domains and without full transitivity of rationalizing relations is well-developed in the literature, these standard notions of rational choice are often considered to be too demanding. An alternative definition of rationality of choice is that of non-deteriorating choice, which requires that the chosen alternatives must be judged at least as good as a reference alternative. In game theory, this definition is well-known under the name of individual rationality when the reference alternative is construed to be the status quo. This alternative form of rationality of individual and social choice is characterized in this paper on general domains and without full transitivity of rationalizing relations.

0. Introduction

The traditional concept of rationality of choice requires the existence of a preference relation on the set of alternatives such that, for any feasible set of options a decision maker may face, the set of chosen options is given by the set of greatest or maximal elements in terms of this preference relation. The origin of this field of *rational choice and revealed preference* can be traced back to consumer theory; see, for example, Samuelson (1938; 1947, Chapter V; 1948; 1950), Houthakker (1950) and Uzawa (1971). In contrast, authors such as Uzawa (1957), Arrow (1959), Sen (1971) and Schwartz (1976) have examined choice situations that do not exhibit the structure that commodity spaces are endowed with; instead, they considered more abstract situations where all finite subsets of a given universal set may appear as feasible sets. The most flexible approach, however, operates with arbitrary domains where no restrictions whatsoever (other than non-emptiness) are imposed on the set of choice situations that we may observe, and focuses on the logic of rationality of choice *per se*. This general model of rational choice has been examined thoroughly in contributions by Richter (1966; 1971), Hansson (1968), Suzumura (1976a; 1977; 1983, Chapter 2) and, more recently, in Bossert, Sprumont and Suzumura (2005a; 2005b; 2006) and in Bossert and Suzumura (2007).

Although greatest-element rationalizability and maximal-element rationali-

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zability are based on a sound normative foundation, these notions of rational choice are sometimes considered too demanding, especially in social choice problems. The requirement that *all* elements of a feasible set should be weakly dominated by a chosen alternative may be rather difficult to satisfy in some circumstances and, thus, we may not necessarily want to declare an agent violating this requirement *irrational*. For instance, the surveys by Camerer (1994) and Shafir and Tversky (1995) report some systematic violations of the standard revealed preference axioms in experimental settings. Thus, it is of interest to examine more modest rationality notions. One such possibility is to analyze the requirement of *non-deteriorating choice,* introduced by Bossert and Sprumont (2007).

The concept of non-deteriorating choice is based on the idea that a chosen alternative need not dominate all elements in the feasible set from which it is chosen but, instead, should be at least as good as a *reference alternative.* This requirement is what is often referred to as *individual rationality* but we prefer to use the term non-deteriorating choice to avoid confusion with rationality defined in terms of best or maximal elements discussed above. To incorporate this idea in a model of choice, we assume that, in each feasible set, there exists a reference alternative which, along with the set of feasible options, determines the choice of the agent. Thus, we work with a *reference-dependent choice function.* In contrast to a traditional choice function which selects a subset of options from each feasible set in its domain, a reference-dependent choice function assigns subsets of chosen options to *pairs,* each of which consists of a feasible set and an alternative belonging to it. There are several natural and plausible interpretations of such a reference alternative. In general, it can be thought of as an alternative representing the *status quo.* This interpretation is applicable in abstract choice problems (which are the ones we focus on here) but also in more specific contexts. For example, in dynamic environments in which consecutive choices have to be made, a plausible reference alternative at a given stage of the process is one that has been selected in the previous stage, provided that it is still feasible in the current stage. In an economic environment, a natural choice of a reference alternative is an initial consumption bundle held by an agent. The potential importance of a status quo alternative is recognized in other contributions as well; see, for instance, Zhou (1997) for an alternative notion of rationality in such a setting. The impact of a reference alternative on choice behaviour is also examined in Rubinstein and Zhou (1999). Masatlioglu and Ok (2005) analyze the notion of a status-quo *bias* in a similar framework.

Non-deterioration can also be examined in a multi-agent setting. In that case, the issue is not merely the existence of a *single* preference relation rationalizing the observed choice in some sense but, rather, the existence of a *profile* of preference relations that generates the observed behaviour in accordance with some theory of collective choice. In non-cooperative settings, Sprumont (2000) examines necessary and sufficient conditions for *Nash rationalizability,* requiring the existence of a profile of preferences defined on combinations of the players' actions such that, for each game defined by a set of feasible actions and the restriction of these preferences to the associated combinations of feasible actions, the set of observations corresponds to the set of Nash equilibria of the game. See Ray and Zhou (2001) for a similar study regarding subgame-perfect equilibria.

In Bossert and Sprumont (2003), two notions that play a fundamental role in essentially all *cooperative* approaches to collective choices are examined. These are (Pareto) efficiency and the above-mentioned notion of non-deterioration. *Efficient and non-deteriorating choice* requires the existence of a profile of preference relations, one for each agent, according to which all selected alternatives are Pareto efficient and at least as good as the reference alternative for every agent.

The existing literature on non-deteriorating choice has focused on *transitive* relations as rationalizations, both in the single-agent setting and in collective choice situations. In this paper, we examine the consequences of weakening this requirement to alternative coherence properties and of dropping it altogether. In that sense, our contribution parallels Bossert, Sprumont and Suzumura (2005a; 2005b; 2006), where maximal-element rationalizability and greatest-element rationalizability by relations that are not necessarily fully transitive are explored.

After introducing our basic definitions, we examine logical relationships between notions of non-deteriorating choice that are obtained by combining coherence properties such as transitivity with one, both, or none of the richness properties of reflexivity and completeness. In particular, we consider definitions of non-deteriorating choice based on transitive, quasi-transitive, consistent and acyclical preferences. Consistency, introduced by Suzumura (1976b), rules out the existence of preference cycles involving at least one strict preference. See Bossert (2007) for a survey of some recent applications of consistency.

In Section 1, we introduce our notions of non-deteriorating choice, followed by an analysis of their logical relationships. In addition, we characterize all distinct forms of non-deterioration. Section 2 provides parallel results for the multi-agent setting.

1. Non-deteriorating Choice

Consider a non-empty (but otherwise arbitrary) set of alternatives X and let $\mathcal{X}$ be the set of all non-empty subsets of X. Let $R \subseteq X \times X$ be a (binary) relation on X. The *asymmetric factor* $P(R)$ of R is defined by

$$P(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \notin R\}.$$

The *symmetric factor* $I(R)$ of R is defined by

$$I(R) = \{(x, y) \in X \times X \mid (x, y) \in R \text{ and } (y, x) \in R\}.$$

The *non-comparable factor* $N(R)$ of R is defined by

$$N(R) = \{(x, y) \in X \times X \mid (x, y) \notin R \text{ and } (y, x) \notin R\}.$$

If R is interpreted as a *weak preference relation,* that is, $(x, y) \in R$ means that x is considered at least as good as y, $P(R)$, $I(R)$ and $N(R)$ can be interpreted as the *strict preference relation*, the *indifference relation* and the *non-comparability relation* corresponding to R, respectively. The *diagonal relation* on X is given by $\Delta = \{(x, x) \mid x \in X\}$.

The *transitive closure* $tc(R)$ of a relation R on X is defined as

$$\begin{aligned} tc(R) \quad = \quad & \{(x, y) \in X \times X \mid \exists K \in \mathbb{N} \text{ and } x^0, \ldots, x^K \in X \text{ such that} \\ & x = x^0, (x^{k-1}, x^k) \in R \text{ for all } k \in \{1, \ldots, K\} \text{ and } x^K = y\}. \end{aligned}$$

Clearly, a relation R is transitive if and only if $R = tc(R)$. The crucial importance of the transitive closure $tc(R)$ is its property of being the smallest transitive relation containing R.

The following properties of a binary relation R are of importance in this paper.

Reflexivity. For all $x \in X$,

$$(x, x) \in R.$$

Completeness. For all $x, y \in X$ such that $x \neq y$,

$$(x, y) \in R \text{ or } (y, x) \in R.$$

Antisymmetry. For all $x, y \in X$,

$$(x, y) \in I(R) \;\Rightarrow\; x = y.$$

Asymmetry. $I(R) = \emptyset$.

Transitivity. For all $x, y, z \in X$,

$$[(x, y) \in R \text{ and } (y, z) \in R] \;\Rightarrow\; (x, z) \in R.$$

Quasi-transitivity. For all $x, y, z \in X$,

$$[(x, y) \in P(R) \text{ and } (y, z) \in P(R)] \;\Rightarrow\; (x, z) \in P(R).$$

Consistency. For all $x, y \in X$,

$$(x, y) \in tc(R) \;\Rightarrow\; (y, x) \notin P(R).$$

Acyclicity. For all $x, y \in X$,

$$(x, y) \in tc(P(R)) \;\Rightarrow\; (y, x) \notin P(R).$$

A reflexive and transitive relation is called a *quasi-ordering* and a complete quasi-ordering is called an *ordering.*

We refer to reflexivity and completeness as *richness* conditions because these two properties require that, at least, some pairs must belong to the relation. In the case of reflexivity, all pairs of the form (x, x) are required to be in the relation, whereas completeness demands that, for any two distinct alternatives x and y, at least one of (x, y) and (y, x) must be in R. Clearly, the reflexivity requirement is equivalent to the set inclusion $\Delta \subseteq R$.

Antisymmetry requires that the relation R be a strict preference relation in the sense that no two *distinct* alternatives can be considered indifferent; it does, however, permit an alternative to be indifferent to itself and, thus, the property is not in conflict with reflexivity. In contrast, asymmetry does not permit any indifference. For example, the asymmetric factor $P(R)$ of a relation R is asymmetric, hence its name.

Transitivity, quasi-transitivity, consistency and acyclicity are *coherence* properties. They require that if certain pairs belong to R, then certain other pairs must belong to R as well (as is the case for transitivity and quasi-transitivity) or certain other pairs cannot belong to R (which applies to the case of consistency and acyclicity). Quasi-transitivity and consistency are independent. A transitive relation is quasi-transitive, and a quasi-transitive relation is acyclical. Moreover, a transitive relation is consistent, and a consistent relation is acyclical. The reverse implications are not true in general. However, the discrepancy between transitivity and consistency disappears if the relation is reflexive and complete; see Suzumura (1983, 244).

Some of the arguments employed in our proofs require the *axiom of choice,* defined as follows.

Axiom of Choice. Suppose that $\mathcal{T}$ is a collection of non-empty sets. Then there exists a function $\varphi\colon \mathcal{T} \to \cup_{T\in\mathcal{T}} T$ such that $\varphi(T) \in T$ for all $T \in \mathcal{T}$.

An *extension* of a relation R is a relation R' such that $R \subseteq R'$ and $P(R) \subseteq P(R')$. If R' is an ordering, it is called an *ordering extension* of R.

A classical theorem due to Szpilrajn (1930) establishes that any asymmetric and transitive relation has an asymmetric, transitive and complete extension. As an immediate consequence of this fundamental theorem, any antisymmetric and transitive relation has an antisymmetric ordering extension. See also Arrow (1951, 64), Hansson (1968) and Suzumura (1976b; 2004) for variants and generalizations of Szpilrajn's theorem.

Theorem 1 *Any antisymmetric and transitive relation R on X has an antisymmetric ordering extension.*

Proof. Suppose R is antisymmetric and transitive. Let $\hat{R} = R \setminus \Delta$. Clearly, $\hat{R}$ is asymmetric and transitive. By Szpilrajn's (1930) theorem, there exists an asymmetric, transitive and complete extension $\hat{R}'$ of $\hat{R}$. Letting $R' = \hat{R}' \cup \Delta$, it follows immediately that R' is an antisymmetric ordering extension of R. ■

Clearly, in order to formulate a precise definition of the concept of non-deterioration, we need to identify the reference alternative for each feasible set and, therefore, a traditional choice function that maps feasible sets into sets of chosen objects is not an adequate description of a choice situation of that nature. To accommodate the presence of a reference alternative, we introduce the notion of a *reference-dependent choice function.* Let $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X} \text{ and } y \in S\}$ be a non-empty domain. The interpretation of the elements in Σ is straightforward: they represent all observable choice situations where, for any $(S, y) \in \Sigma$, S is the feasible set of options and $y \in S$ is the reference alternative. A reference-dependent choice function is a mapping $C\colon \Sigma \to \mathcal{X}$ such that, for all $(S, y) \in \Sigma$, $C(S, y) \subseteq S$. The image of C is $C(\Sigma) = \cup_{(S,y)\in\Sigma} C(S, y)$.

In line with the intuitive interpretation of non-deteriorating choice given above, we say that a reference-dependent choice function C is *ND-rationalizable* if and only if there exists an antisymmetric relation R on X such that, for all $(S, y) \in \Sigma$ and for all $x \in C(S, y)$,

$$(x, y) \in R.$$

A relation R with this property is said to be an *ND-rationalization* of C or, alternatively, C is *ND-rationalized* by R.

The antisymmetry assumption imposed on R is intended to avoid degenerate situations. Without a restriction such as antisymmetry, the concept of non-deterioration becomes vacuous: *any* reference-dependent choice function would be declared to be ND-rationalizable if we were to permit the *universal indifference* relation—the relation $R = X \times X$—as a potential ND-rationalization. Although representing a rather weak notion of rationality even if antisymmetry is imposed on an ND-rationalization, non-deterioration as defined above is not a vacuous concept. For instance, suppose that $X = \{x, y\}$, $\Sigma = \{(\{x, y\}, x), (\{x, y\}, y)\}$, $C(\{x, y\}, x) = \{y\}$ and $C(\{x, y\}, y) = \{x\}$. Clearly, ND-rationalizability requires the existence of an antisymmetric relation R such that $(x, y) \in R$ and $(y, x) \in R$ which is an immediate contradiction to the antisymmetry assumption.

Depending on whether one, both or none of the two richness properties and one or none of the four coherence requirements are imposed in addition to ND-rationalizability, we obtain different versions of rationalizability in the sense of non-deteriorating choice. For simplicity of presentation, we use the following convention when identifying a rationalizability axiom. ND-rationalizability is abbreviated by **ND**, **R** stands for reflexivity and **C** is completeness. Transitivity, quasi-transitivity, consistency and acyclicity are denoted by **T**, **Q**, **S** and **A**, respectively. If none of the properties is required, this is denoted by using the symbol $\emptyset$. Formally, a rationalizability property is identified by an expression of the form **ND-**β**-**γ, where $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \emptyset\}$ and $\gamma \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \emptyset\}$. For example, ND-rationalizability by a reflexive, complete and transitive relation is denoted by **ND-RC-T**, ND-rationalizability by a complete relation is **ND-C-**$\emptyset$, ND-rationalizability by a reflexive and consistent relation is **ND-R-S** and ND-rationalizability without any further properties of a rationalizing relation is **ND-**$\emptyset$**-**$\emptyset$.

There are, in principle, $4 \cdot 5 = 20$ versions of ND-rationalizability according to this classification. It turns out, however, that there remain merely two distinct ones because many of them are equivalent, even on arbitrary domains. This is in stark contrast with the results obtained for greatest-element rationalizability and maximal-element rationalizability, where eleven and four distinct versions, respectively, can be identified; see Bossert and Suzumura (2007) for details. Thus, we can think of this notion as being remarkably robust with respect to the additional properties that are imposed on an ND-rationalization.

Before stating a formal result regarding the logical relationships between all of our notions of ND-rationalizability, we provide a preliminary observation which is analogous to the relationship between the direct revealed preference relation of a choice function and any greatest-element rationalization thereof. Analogously to the direct revealed preference relation associated with a traditional choice function, we define the relation R_C corresponding to a reference-dependent choice function $C\colon \Sigma \to \mathcal{X}$ by

$$R_C = \{(x, y) \in X \times X \mid \exists S \in \mathcal{X} \text{ such that } (S, y) \in \Sigma \text{ and } x \in C(S, y)\}.$$

As in the case for the direct revealed preference relation of a standard choice function, any ND-rationalization of a reference-dependent choice function C must respect the relation R_C.

Theorem 2 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X}$ and $y \in S\}$ and R is a relation on X. If R is an ND-rationalization of C, then $R_C \subseteq R$.*

Proof. Suppose that R is an ND-rationalization of C and $x, y \in X$ are such that $(x, y) \in R_C$. By definition of R_C, there exists $S \in \mathcal{X}$ such that $(S, y) \in \Sigma$ and $x \in C(S, y)$. Because R is an ND-rationalization of C, this implies $(x, y) \in R$. ■

The following theorem shows that the only distinction to be made between our different notions of ND-rationalizability is whether an ND-rationalization possesses any of the coherence properties of transitivity, consistency, quasi-transitivity or acyclicity: as soon as one of these conditions is satisfied, all of them are. Moreover, both reflexivity and completeness are redundant because any notion of ND-rationalizability without these richness properties is equivalent to that obtained by adding both of them. For convenience, we employ the following diagrammatic representation throughout the paper. All axioms that are depicted within the same box are equivalent, and an arrow pointing from one box b to another box b' indicates that the axioms in b imply those in b', and the converse implication is not true. In addition, of course, all implications resulting from chains of arrows depicted in such a diagram are valid.

Theorem 3 *Suppose $C: \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S,y) \mid S \in \mathcal{X}$ and $y \in S\}$. Then*

ND-RC-T, **ND-C-T**, **ND-R-T**, **ND-∅-T**, **ND-RC-S**, **ND-C-S**, **ND-R-S**, **ND-∅-S**, **ND-RC-Q**, **ND-C-Q**, **ND-R-Q**, **ND-∅-Q**, **ND-RC-A**, **ND-C-A**, **ND-R-A**, **ND-∅-A**

$\downarrow$

ND-RC-∅, **ND-C-∅**, **ND-R-∅**, **ND-∅-∅**

Proof. To establish the theorem, we need to show that the properties in each of the two boxes are equivalent and, furthermore, that the implication indicated by the arrow in the theorem statement is strict; it is obvious that the implication itself is true.

(a) We first prove the equivalence of the axioms in the top box. To do so, it clearly is sufficient to show that **ND-∅-A** implies **ND-RC-T**. Suppose R is an acyclical ND-rationalization of C. Consider the transitive relation $tc(R) \cup \Delta$. We first prove that $tc(R) \cup \Delta$ is antisymmetric. By way of contradiction, suppose this is not the case. Then there exist $x, y \in X$ such that $(x, y) \in I(tc(R) \cup \Delta)$ and $x \neq y$. Because $x \neq y$, $(x, y) \notin \Delta$ and, thus, by definition of R, there exist $K, L \in \mathbb{N}$ and $x^0, \ldots, x^K, y^0, \ldots, y^L \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$, $x^K = y = y^0$, $(y^{\ell-1}, y^\ell) \in R$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = x$. Clearly, we can, without loss of generality, assume that the x^k are pairwise distinct and, analogously, the y^ℓ are pairwise distinct. Because R is an ND-rationalization of C and, thus, antisymmetric, it follows that $(x^{k-1}, x^k) \in P(R)$ for all $k \in \{1, \ldots, K\}$ and $(y^{\ell-1}, y^\ell) \in P(R)$ for all $\ell \in \{1, \ldots, L\}$. But this contradicts the acyclicity of R. Therefore, R is antisymmetric.

By Theorem 1, $tc(R) \cup \Delta$ has an antisymmetric ordering extension R'. To complete the proof that R' is an ND-rationalization of C, suppose that $x \in X$ and $(S, y) \in \Sigma$ are such that $x \in C(S, y)$. By definition of R_C, this implies $(x, y) \in R_C$. Using Theorem 2, the definition of the transitive closure of a relation and the definition of R', we have $R_C \subseteq R \subseteq tc(R) \subseteq tc(R) \cup \Delta \subseteq R'$. Thus, $(x, y) \in R'$.

(b) To prove the equivalence of the axioms in the second box, it suffices to show that **ND-∅-∅** implies **ND-RC-∅**. Let R be an ND-rationalization of C.

If R is complete, the relation $R' = R \cup \Delta$ clearly is a reflexive and complete ND-rationalization of C.

Now suppose R is not complete. Let $\mathcal{T} = \{\{x, y\} \mid (x, y) \in N(R)$ and $x \neq y\}$. Because R is not complete, it follows that $\mathcal{T} \neq \emptyset$. By the axiom of choice, there exists a function $\varphi: \mathcal{T} \to \cup_{T \in \mathcal{T}} T$ such that $\varphi(T) \in T$ for all $T \in \mathcal{T}$. Let

$$R' = R \cup \Delta \cup \{(\varphi(\{x, y\}), z) \mid \{x, y\} \in \mathcal{T} \text{ and } \{x, y\} \setminus \{\varphi(\{x, y\})\} = \{z\}\}.$$

Clearly, R' is reflexive and complete. To see that R' is antisymmetric, note that the three relations, the union of which constitutes R', are antisymmetric and the relation

$$\{(\varphi(\{x,y\}),z) \mid \{x,y\} \in \mathcal{T} \text{ and } \{x,y\} \setminus \{\varphi(\{x,y\})\} = \{z\}\}$$

only contains pairs of distinct elements that are non-comparable according to $R \cup \Delta$. To complete the proof that R' is an ND-rationalization of C, suppose $x \in X$ and $(S,y) \in \Sigma$ are such that $x \in C(S,y)$. Because R is an ND-rationalization of C, it follows that $(x,y) \in R$ and, because $R \subseteq R'$ by definition, $(x,y) \in R'$.

(c) To see that the implication in the theorem statement is strict, consider the following example. Let $X = \{x,y,z\}$ and $\Sigma = \{(S,w) \mid S \in \mathcal{X} \text{ and } w \in S\}$, and define a reference-dependent choice function $C\colon \Sigma \to \mathcal{X}$ by $C(\{x,y\},y) = \{x\}$, $C(\{x,z\},x) = \{z\}$, $C(\{y,z\},z) = \{y\}$ and $C(S,w) = \{w\}$ for all $(S,w) \in \Sigma \setminus \{(\{x,y\},y),(\{x,z\},x),(\{y,z\},z)\}$. This reference-dependent choice function is ND-rationalized by the (antisymmetric) relation

$$R = \{(x,x),(x,y),(y,y),(y,z),(z,x),(z,z)\}$$

and, therefore, C satisfies **ND-$\emptyset$-$\emptyset$**. By way of contradiction, suppose that C satisfies **ND-$\emptyset$-A** and let R' be an acyclical ND-rationalization of C. It follows that we must have $(x,y) \in R'$ because $x \in C(\{x,y\},y)$, $(y,z) \in R'$ because $y \in C(\{y,z\},z)$ and $(z,x) \in R'$ because $z \in C(\{x,z\},x)$. Because ND-rationalizability requires that R' is antisymmetric, it follows that $(x,y) \in P(R')$, $(y,z) \in P(R')$ and $(z,x) \in P(R')$, contradicting the acyclicity of R'. ■

The reference-dependent choice function employed in Part (c) of the above proof is defined on the full domain $\{(S,y) \mid S \in \mathcal{X} \text{ and } y \in S\}$. Therefore, the logical relationships displayed in the theorem statement (in particular, the *strict* implication) remain true even if the domain Σ is assumed to be extremely rich.

We now provide characterizations of the two distinct notions of ND-rationalizability identified in the above theorem. We begin with a characterization of the properties in the top box of Theorem 3, which is due to Bossert and Sprumont (2007). Analogously to the congruence axiom of Richter (1966), we define a variant that is suitable for reference-dependent choice functions.

Reference-dependent congruence. For all $S \in \mathcal{X}$ and for all $x,y \in X$,

$$[(x,y) \in tc(R_C) \text{ and } (S,x) \in \Sigma \text{ and } x \neq y] \Rightarrow y \notin C(S,x).$$

As in the case of the congruence axiom defined for choice functions, reference-dependent congruence ensures that chains of preference according to R_C are respected. This axiom is necessary and sufficient for ND-rationalizability on any domain.

Theorem 4 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S,y) \mid S \in \mathcal{X}$ and $y \in S\}$. C satisfies* **ND-**β**-**γ *for any $\beta \in \{\mathbf{RC},\mathbf{R},\mathbf{C},\emptyset\}$ and any $\gamma \in \{\mathbf{T},\mathbf{Q},\mathbf{S},\mathbf{A}\}$ if and only if C satisfies reference-dependent congruence.*

Proof. By Theorem 3, it is sufficient to establish the equivalence of **ND-∅-T** and reference-dependent congruence.

Suppose first that C satisfies **ND-∅-T** and that R is a transitive ND-rationalization of C. By way of contradiction, suppose that reference-dependent congruence is violated. Then there exist $S \in \mathcal{X}$ and $x, y \in X$ such that $(x, y) \in tc(R_C)$, $(S, x) \in \Sigma$, $x \neq y$ and $y \in C(S, x)$. By definition, $(y, x) \in R_C$ and, by Theorem 2, $(y, x) \in R$. Because $(x, y) \in tc(R_C)$, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. Using Theorem 2 again, we obtain $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$ and, because R is transitive, $(x, y) \in R$. Because $x \neq y$ by assumption, this contradicts the antisymmetry of R.

Now suppose C satisfies reference-dependent congruence. We complete the proof by establishing that the transitive relation $R = tc(R_C)$ is an ND-rationalization of C. To show that $R = tc(R_C)$ is antisymmetric, suppose, to the contrary, that there exist $x, y \in X$ such that $x \neq y$ and $(x, y) \in I(R) = I(tc(R_C))$. Then there exist $K, L \in \mathbb{N}$, $x^0, \ldots, x^K \in X$ and $y^0, \ldots, y^L \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R$ for all $k \in \{1, \ldots, K\}$, $x^K = y = y^0$, $(y^{\ell-1}, y^\ell) \in R$ for all $\ell \in \{1, \ldots, L\}$ and $y^L = x$. Clearly, we can, without loss of generality, assume that the x^k are pairwise distinct and that the y^ℓ are pairwise distinct. Thus, $y^{L-1} \neq y^L = x$. By definition, $(x, y^{L-1}) \in tc(R_C)$ and there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y^{L-1} \in C(S, x)$, contradicting reference-dependent congruence. To complete the proof, note that $x \in C(S, y)$ for some $x \in X$ and $(S, y) \in \Sigma$ immediately implies $(x, y) \in R_C \subseteq tc(R_C) = R$. ■

Our next task is the characterization of the remaining four (equivalent) notions of ND-rationalizability. The following axiom of weak reference-dependent congruence is obtained from reference-dependent congruence by replacing the transitive closure of R_C with R_C itself.

Weak reference-dependent congruence. For all $S \in \mathcal{X}$ and for all $x, y \in X$,

$$[(x, y) \in R_C \text{ and } (S, x) \in \Sigma \text{ and } x \neq y] \Rightarrow y \notin C(S, x).$$

Interestingly, whereas the analogous weakening of Richter's (1966) congruence axiom is not a necessary and sufficient condition for the greatest-element rationalizability of a traditional choice function (see Richter, 1971), weak reference-dependent congruence can be used to provide a characterization of the ND-rationalizability of a reference-dependent choice function in the absence of any coherence property.

Theorem 5 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X} \text{ and } y \in S\}$. C satisfies* **ND-**β*–∅ for any $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \emptyset\}$ if and only if C satisfies weak reference-dependent congruence.*

Proof. By Theorem 3, it is sufficient to establish the equivalence of **ND-∅-**∅ and weak reference-dependent congruence.

Suppose first that C satisfies **ND-∅-∅** and that R is an ND-rationalization of C. By way of contradiction, suppose that weak reference-dependent congruence is violated. Then there exist $S \in \mathcal{X}$ and $x, y \in X$ such that $(x, y) \in R_C$, $(S, x) \in \Sigma$, $x \neq y$ and $y \in C(S, x)$. By definition, $(y, x) \in R_C$ and, by Theorem 2, $(y, x) \in R$. Invoking Theorem 2 again, the assumption $(x, y) \in R_C$ implies $(x, y) \in R$ and, because $x \neq y$, we obtain a contradiction to the antisymmetry of R.

Now suppose C satisfies weak reference-dependent congruence. We prove that $R = R_C$ is an ND-rationalization of C. To show that $R = R_C$ is antisymmetric, suppose, to the contrary, that there exist $x, y \in X$ such that $x \neq y$ and $(x, y) \in I(R) = I(R_C)$. Because $(y, x) \in R_C$, there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y \in C(S, x)$. Thus, we have $(x, y) \in R_C$, $(S, x) \in \Sigma$, $x \neq y$ and $y \in C(S, x)$, contradicting weak reference-dependent congruence. To complete the proof, note that $x \in C(S, y)$ for some $x \in X$ and $(S, y) \in \Sigma$ immediately implies $(x, y) \in R_C$. ■

2. Efficient and Non-deteriorating Choice

We now move on to a discussion of non-deteriorating choice in a multi-agent environment. In the case of theories of *collective* choice, a test of a particular theory (or a class of theories) involves not only a *single* relation that rationalizes the observed choices according to a particular notion of rationalizability but, instead, an entire *profile* of preference relations, one relation for each member of society.

Our approach follows that of Bossert and Sprumont (2003) which focuses on two central features of *cooperative* collective choice. In addition to requiring reference-dependent choices to be non-deteriorating for each agent, they must be *efficient* according to the rationalizing profile of individual preferences. Both of these properties are of fundamental importance in many applications; for instance, studying non-deteriorating and efficient behaviour is essential in developing testable restrictions of prominent concepts such as the core and the set of Walrasian equilibria in an exchange economy; see, for instance, Brown and Matzkin (1996) and Bossert and Sprumont (2002). As we do throughout this paper, we will, however, focus on abstract choice problems in order to provide the most general treatment.

Suppose there is a set $\{1, \ldots, n\}$ of $n \in \mathbb{N} \setminus \{1\}$ agents. A reference-dependent choice function C on an arbitrary domain is defined as in the single-agent case discussed in the previous section. Analogously, the definition of the relation R_C is unchanged. However, the notion of rationalizability we examine now differs from that of ND-rationalizability: instead of merely considering non-deteriorating choice, we now analyze *efficient and non-deteriorating* choice. We say that a reference-dependent choice function C is *E-rationalizable* if and only if there exists a profile $(R_1, \ldots, R_n)$ of antisymmetric relations on X such that, for all $(S, y) \in \Sigma$, for all $x \in C(S, y)$ and for all $i \in \{1, \ldots, n\}$,

$$(x, y) \in R_i \tag{1}$$

and, for all $(S, y) \in \Sigma$ and for all $x \in C(S, y)$,

$$\{z \in S \mid (z, x) \in P(R_i) \text{ for all } i \in \{1, \ldots, n\}\} = \emptyset. \tag{2}$$

A profile of antisymmetric relations with these properties is said to be an *E-rationalization* of C or, alternatively, C is *E-rationalized* by the profile $(R_1, \ldots, R_n)$. R_i being antisymmetric for all $i \in \{1, \ldots, n\}$, (2) can be equivalently written as follows:

$$\{z \in S \setminus \{x\} \mid (z, x) \in R_i \text{ for all } i \in \{1, \ldots, n\}\} = \emptyset \tag{3}$$

The efficiency requirement (2) or, equivalently, (3) by itself does not impose any restrictions. For any reference-dependent choice function C, let R_1 be an arbitrary antisymmetric relation and define

$$R_i = \{(x, y) \mid (y, x) \in R_1\}$$

for all $i \in \{2, \ldots, n\}$, that is, each R_i with $i \in \{2, \ldots, n\}$ is given by the *inverse* of R_1. Clearly, *all* elements of X are efficient for this profile and, thus, (3) is satisfied for any reference-dependent choice function C.

Interestingly, when combined with the non-deterioration requirement (1), efficiency *does* impose further restrictions. For example, let $X = \{x, y, z\}$, $\Sigma = \{(X, y), (X, z)\}$, $C(X, y) = \{x\}$ and $C(X, z) = \{y\}$. Clearly, there exists a profile of antisymmetric relations $(R_1, \ldots, R_n)$ such that (1) is satisfied (letting, for instance, $R_i = \{(x, y), (y, z)\}$ for all $i \in \{1, \ldots, n\}$ will do) but any such profile must be such that $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$ because $x \in C(X, y)$. But this contradicts efficiency because $x \in X \setminus \{y\}$ and $y \in C(X, z)$.

We now analyze the possible notions of E-rationalizability that are obtained by adding our combinations of richness and coherence properties. For $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \emptyset\}$ and $\gamma \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \emptyset\}$, **E-**$\beta$**-**$\gamma$ denotes E-rationalizability by a relation satisfying the richness property or properties represented by β and the coherence property identified by γ.

As a preliminary observation, we note that any E-rationalization $(R_1, \ldots, R_n)$ of a reference-dependent choice function C must be such that all relations R_i respect the relation R_C; this result is parallel to Theorem 2. Efficiency is not required for this implication—it is sufficient to assume that (1) is satisfied.

Theorem 6 *Suppose* $C\colon \Sigma \to \mathcal{X}$ *is a reference-dependent choice function with an arbitrary non-empty domain* $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X}$ *and* $y \in S\}$ *and* $(R_1, \ldots, R_n)$ *is a profile of antisymmetric relations on* X*. If* C *and* $(R_1, \ldots, R_n)$ *are such that (1) is satisfied, then* $R_C \subseteq \bigcap_{i=1}^n R_i$*.*

Proof. Suppose that C and $(R_1, \ldots, R_n)$ are such that (1) is satisfied and $x, y \in X$ are such that $(x, y) \in R_C$. By definition of R_C, there exists $S \in \mathcal{X}$ such that $(S, y) \in \Sigma$ and $x \in C(S, y)$. By (1), this implies $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$. ■

Analogously to our procedure employed for the single-agent case, we now examine the logical relationships between the various notions of E-rationalizability on arbitrary domains. As a preliminary observation, note that the distinction between transitivity and quasi-transitivity disappears in the presence of antisymmetry and, analogously, consistency and acyclicity are equivalent for antisymmetric relations. We provide a formal statement of this result for future reference but do not present the straightforward proof.

Theorem 7 *Suppose R is an antisymmetric relation on X.*

(i) If R is quasi-transitive, then R is transitive.

(ii) If R is acyclical, then R is consistent.

We now prove that, out of a possible twenty, only four distinct versions of E-rationalizability exist.

Theorem 8 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X}$ and $y \in S\}$. Then*

E-RC-T, **E-C-T**, **E-RC-S**, **E-C-S**,
E-RC-Q, **E-C-Q**, **E-RC-A**, **E-C-A**

↓

E-R-T, **E-∅-T**, **E-R-Q**, **E-∅-Q**

↓

E-R-S, **E-∅-S**, **E-R-A**, **E-∅-A**

↓

E-RC-∅, **E-C-∅**, **E-R-∅**, **E-∅-∅**

Proof. The implications in the theorem statement are straightforward. Thus, it remains to establish the equivalences in each of the four boxes and to provide three examples showing that the implications are strict.

(a) By Theorem 7, **E-RC-A** and **E-RC-S** are equivalent. Furthermore, because consistency and transitivity coincide in the presence of reflexivity and completeness, **E-RC-S** and **E-RC-T** are equivalent. Thus, in order to establish the equivalences in the first box, it is sufficient to prove that **E-C-A** implies **E-RC-A**. Let $(R_1, \ldots, R_n)$ be an E-rationalization of C such that the R_i are complete and acyclical. Letting $R_i' = R_i \cup \Delta$ for all $i \in \{1, \ldots, n\}$, it is immediate that each R_i' is reflexive, complete and acyclical and $(R_1', \ldots, R_n')$ is an E-rationalization of C.

(b) Given Part (i) of Theorem 7, the equivalence of the axioms in the second box follows as soon as we establish that **E-∅-Q** implies **E-R-Q**. As in (a), it is straightforward to see that if $(R_1, \ldots, R_n)$ is an E-rationalization of C and the R_i are quasi-transitive, then $(R_1 \cup \Delta, \ldots, R_n \cup \Delta)$ is an E-rationalization of C and $R_i \cup \Delta$ is reflexive and quasi-transitive for all $i \in \{1, \ldots, n\}$.

(c) Analogously, the equivalence of the axioms in the third box follows from Part (ii) of Theorem 7 and the observation that if $(R_1, \ldots, R_n)$ is an E-rationalization of C with acyclical relations R_i, then $(R_1 \cup \Delta, \ldots, R_n \cup \Delta)$

is an E-rationalization of C such that $R_i \cup \Delta$ is reflexive and acyclical for all $i \in \{1, \ldots, n\}$.

(d) To show that the axioms in the last box are equivalent, it is sufficient to prove that **E-∅-∅** implies **E-RC-∅**. Suppose $(R_1, \ldots, R_n)$ is an E-rationalization of C.

If R_i is complete for all $i \in \{1, \ldots, n\}$, the profile $(R'_1, \ldots, R'_n)$ obtained by letting $R'_i = R_i \cup \Delta$ for all $i \in \{1, \ldots, n\}$ clearly is an E-rationalization of C and the R'_i are reflexive and complete.

Now suppose there exists $i \in \{1, \ldots, n\}$ such that R_i is not complete. Let $\mathcal{N} \subseteq \{1, \ldots, n\}$ be the set of individuals $i \in \{1, \ldots, n\}$ such that R_i is incomplete. Clearly, $\mathcal{N}$ is non-empty by assumption, and it may coincide with the entire set $\{1, \ldots, n\}$. Let $m = |\mathcal{N}| \in \{1, \ldots, n\}$ be the number of elements in $\mathcal{N}$ and suppose, without loss of generality, that $\mathcal{N} = \{1, \ldots, m\}$. Define $R'_j = R_j \cup \Delta$ for all $j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$. Clearly, R'_j is reflexive, complete and antisymmetric for all $j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$.

Let

$$R_1^0 = R_1 \cup \Delta \cup \{(x, y) \in N(R_1) \mid (y, x) \in R'_j \text{ for all } j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\} \text{ and } x \neq y\}.$$

If R_1^0 is complete, let $R'_1 = R_1^0$. It is evident that R'_1 is reflexive and complete. To establish that R'_1 is antisymmetric, observe first that R_1 is antisymmetric by assumption and Δ is trivially antisymmetric. Furthermore, if $(y, x) \in R'_j$ for all $j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ for any two distinct alternatives x and y, then the antisymmetry of the relations R'_j implies that we cannot have $(x, y) \in R'_j$ for all $j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$ and, thus, the relation $\{(x, y) \in N(R_1) \mid (y, x) \in R'_j \text{ for all } j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\} \text{ and } x \neq y\}$ is antisymmetric as well. Finally, note that the latter relation only contains pairs that are non-comparable according to $R_1 \cup \Delta$.

If R_1^0 is not complete, define $\mathcal{T}_1 = \{\{x, y\} \mid (x, y) \in N(R_1^0)\}$. Because R_1^0 is not complete, it follows that $\mathcal{T}_1 \neq \emptyset$. By the axiom of choice, there exists a function $\varphi_1 : \mathcal{T}_1 \to \cup_{T \in \mathcal{T}_1} T$ such that $\varphi_1(T) \in T$ for all $T \in \mathcal{T}_1$. Let

$$R'_1 = R_1^0 \cup \{(\varphi_1(\{x, y\}), z) \mid \{x, y\} \in \mathcal{T}_1 \text{ and } \{x, y\} \setminus \{\varphi_1(\{x, y\})\} = \{z\}\}.$$

Clearly, R'_1 is reflexive and complete. To see that R'_1 is antisymmetric, note that the two relations, the union of which constitutes R'_1, are antisymmetric and that the second of these relations only contains pairs of distinct elements that are non-comparable according to R_1^0.

If $m = 1$, we have defined a profile $(R'_1, \ldots, R'_n)$ of reflexive, complete and antisymmetric relations. If $m > 1$, we define the relations $R'_2, \ldots, R'_m$ iteratively as follows. Let $i \in \{2, \ldots, m\}$, and suppose R'_j has been defined for all $j \in \{1, \ldots, i-1\}$ (note that R'_j is already defined for all $j \in \{1, \ldots, n\} \setminus \{1, \ldots, m\}$). Define

$$R_i^0 = R_i \cup \Delta \cup \{(x, y) \in N(R_i) \mid (y, x) \in R'_j \text{ for all } j \in \{1, \ldots, n\} \setminus \{i, \ldots, m\} \text{ and } x \neq y\}.$$

If R_i^0 is complete, let $R_i' = R_i^0$. The proof that R_i' is reflexive, complete and antisymmetric is identical to that establishing these properties for R_1'.

If R_i^0 is not complete, define $\mathcal{T}_i = \{\{x,y\} \mid (x,y) \in N(R_i^0)\}$. Because R_i^0 is not complete, it follows that $\mathcal{T}_i \neq \emptyset$. By the axiom of choice, there exists a function $\varphi_i: \mathcal{T}_i \to \cup_{T\in\mathcal{T}_i} T$ such that $\varphi_i(T) \in T$ for all $T \in \mathcal{T}_i$. Let

$$R_i' = R_i^0 \cup \{(\varphi_i(\{x,y\}), z) \mid \{x,y\} \in \mathcal{T}_i \text{ and } \{x,y\} \setminus \{\varphi_i(\{x,y\})\} = \{z\}\}.$$

Again, that R_i' is reflexive, complete and antisymmetric follows from the same argument as that employed to establish these properties for R_1'.

We have now defined a profile $(R_1', \ldots, R_n')$ of reflexive, complete and antisymmetric relations. To complete the proof that this profile is an E-rationalization of C, it has to be shown that (1) and (3) are satisfied.

To establish (1), suppose $x \in X$ and $(S,y) \in \Sigma$ are such that $x \in C(S,y)$. Because $(R_1, \ldots, R_n)$ is an E-rationalization of C, it follows that $(x,y) \in R_i$ for all $i \in \{1, \ldots, n\}$ and, because $R_i \subseteq R_i'$ by definition, we obtain $(x,y) \in R_i'$ for all $i \in \{1, \ldots, n\}$.

Finally, we show that (3) is satisfied. By way of contradiction, suppose there exist $(S,y) \in \Sigma$, $x \in C(S,y)$ and $z \in S \setminus \{x\}$ such that $(z,x) \in R_i'$ for all $i \in \{1, \ldots, n\}$. Because $z \neq x$, $(z,x) \notin \Delta$.

If $(z,x) \in R_i$ for all $i \in \{1, \ldots, n\}$, we immediately obtain a contradiction to the assumption that $(R_1, \ldots, R_n)$ is an E-rationalization of C.

If there exists $i \in \{1, \ldots, n\}$ such that $(z,x) \in R_i^0 \setminus (R_i \cup \Delta)$, it follows that $(x,z) \in R_j'$ for all $j \in \{1, \ldots, n\} \setminus \{i, \ldots, m\}$. Because $(z,x) \in R_j'$ for all $j \in \{1, \ldots, n\} \setminus \{i, \ldots, m\}$ by assumption, this contradicts the antisymmetry of these relations.

The last remaining possibility is that there exists $i \in \{1, \ldots, n\}$ such that $(z,x) \in R_i' \setminus R_i^0$. Let $k = \max\ \{i \in \{1, \ldots, n\} \mid (z,x) \in R_i' \setminus R_i^0\}$. Because $(z,x) \notin R_k^0$, it follows that there exists $j \in \{1, \ldots, n\} \setminus \{k, \ldots, m\}$ such that $(z,x) \notin R_j'$, contradicting our hypothesis that $(z,x) \in R_j'$ for all $j \in \{1, \ldots, n\}$.

(e) We now show that the first implication in the theorem statement is strict. To do so, we employ an example used in Bossert and Sprumont (2003) which, in turn, is an adaptation of an example developed in Sprumont (2001) in a different context. Let $X = \{x^1, \ldots, x^{n+1}, y^1, \ldots, y^{n+1}\}$ and

$$\Sigma = \bigcup_{j=1}^{n+1} \left\{(\{x^1, \ldots, x^{n+1}, y^j\}, y^j), (\{x^j, y^j\}, y^j)\right\}.$$

Define a reference-dependent choice function C by

$$C(\{x^1, \ldots, x^{n+1}, y^j\}, y^j) = \{x^1, \ldots, x^{n+1}\} \setminus \{x^j\} \text{ and } C(\{x^j, y^j\}, y^j) = \{y^j\}$$

for all $j \in \{1, \ldots, n+1\}$. C is E-rationalized by the profile $(R_1, \ldots, R_n)$ of transitive relations given by

$$R_i = \left\{(x^j, y^k) \mid j,k \in \{1, \ldots, n+1\} \text{ and } j \neq k\right\} \cup \left\{(y^j, y^j) \mid j \in \{1, \ldots, n+1\}\right\}$$

for all $i \in \{1, \ldots, n\}$ but there exists no E-rationalization of C that is composed of orderings. To verify this fact by way of contradiction, suppose $(R_1', \ldots, R_n')$ is an E-rationalization of C and the R'_i are orderings on X. Because these relations are antisymmetric orderings, it follows that, for each $i \in \{1, \ldots, n\}$, there exists a unique $x_i^* \in \{x^1, \ldots, x^{n+1}\}$ such that $(x^j, x_i^*) \in R'_i$ for all $j \in \{1, \ldots, n+1\}$, that is, x_i^* is the worst element in $\{x^1, \ldots, x^{n+1}\}$ according to R'_i. Because the number of agents n is less than the number of elements $n+1$ in $\{x^1, \ldots, x^{n+1}\}$, there exists at least one $k \in \{1, \ldots, n+1\}$ such that x^k is not the worst element in $\{x^1, \ldots, x^{n+1}\}$ for any of the agents. Thus, we have $(x^k, x_i^*) \in R'_i$ for all $i \in \{1, \ldots, n\}$. By (1), $(x_i^*, y^k) \in R'_i$ for all $i \in \{1, \ldots, n\}$. Transitivity implies $(x^k, y^k) \in R'_i$ for all $i \in \{1, \ldots, n\}$. Because $y^k \in C(\{x^k, y^k\}, y^k)$, this contradicts the efficiency requirement (3).

(f) To show that the second implication is strict, consider the following example. Let $X = \{x, y, z\}$, $\Sigma = \{(\{x, y\}, y), (\{x, z\}, z), (\{y, z\}, z)\}$, $C(\{x, y\}, y) = \{x\}$, $C(\{x, z\}, z) = \{z\}$ and $C(\{y, z\}, z) = \{y\}$. This reference-dependent choice function is E-rationalized by the profile $(R_1, \ldots, R_n)$ of consistent relations defined by

$$R_i = \{(x, y), (y, z), (z, z)\}$$

for all $i \in \{1, \ldots, n\}$. By way of contradiction, suppose $(R'_1, \ldots, R'_n)$ is an E-rationalization of C with transitive relations $R'_1, \ldots, R'_n$. By (1), we must have $(x, y) \in R'_i$ and $(y, z) \in R'_i$ for all $i \in \{1, \ldots, n\}$ because $x \in C(\{x, y\}, y)$ and $y \in C(\{y, z\}, z)$. Transitivity implies $(x, z) \in R'_i$ for all $i \in \{1, \ldots, n\}$, which contradicts (3) because $z \in C(\{x, z\}, z)$ and $x \in \{x, z\} \setminus \{z\}$.

(g) Finally, we show that the last implication is strict. Let $X = \{x, y, z\}$ and $\Sigma = \{(\{x, y\}, y), (\{x, z\}, x), (\{y, z\}, z)\}$, and define a reference-dependent choice function $C\colon \Sigma \to \mathcal{X}$ by $C(\{x, y\}, y) = \{x\}$, $C(\{x, z\}, x) = \{z\}$ and $C(\{y, z\}, z) = \{y\}$. This reference-dependent choice function is E-rationalized by the profile $(R_1, \ldots, R_n)$ such that

$$R_i = \{(x, y), (y, z), (z, x)\}$$

for all $i \in \{1, \ldots, n\}$. By way of contradiction, suppose $(R'_1, \ldots, R'_n)$ is an E-rationalization of C that is composed of acyclical relations. By (1), it follows that we must have, for all $i \in \{1, \ldots, n\}$, $(x, y) \in R'_i$ because $x \in C(\{x, y\}, y)$, $(y, z) \in R'_i$ because $y \in C(\{y, z\}, z)$ and $(z, x) \in R'_i$ because $z \in C(\{x, z\}, x)$. By antisymmetry, it follows that $(x, y) \in P(R'_i)$, $(y, z) \in P(R'_i)$ and $(z, x) \in P(R'_i)$ for all $i \in \{1, \ldots, n\}$, contradicting the acyclicity of the R'_i. ■

Our next task is the characterization of the various notions of E-rationalizability distinguished in the above theorem. We provide characterizations of the three weakest versions. Interestingly, while the previously discussed single-agent versions of ND-rationalizability are relatively straightforward to characterize if the rationalizing relation is required to be an ordering and difficulties typically arise if weaker coherence properties are imposed, the opposite is true for E-rationalizability: except for special cases in which the strongest notion of E-rationalizability coincides with the second-strongest version, the characterization of this property is an open problem. We provide a discussion of this issue following our characterization results.

A condition that is necessary and sufficient for **E-∅-T** and its equivalent properties is obtained by a natural extension of reference-dependent congruence.

Extended reference-dependent congruence. For all $(S, z) \in \Sigma$ and for all $x, y \in X$,

$$[(x, y) \in tc(R_C) \text{ and } x \in S \text{ and } x \neq y] \Rightarrow y \notin C(S, z).$$

In addition to the restrictions imposed by reference-dependent congruence (which is the special case of this axiom obtained for $z = x$), extended reference-dependent congruence requires that if there is a chain of preferences involving distinct elements according to R_C, then the last element in the chain cannot be chosen if the first element of the chain is feasible; this applies even if the first element of the chain is not the reference alternative. Clearly, this additional requirement is imposed by the conjunction of transitivity, non-deterioration and efficiency: transitivity and non-deterioration require that this chain of preferences be respected by all individual relations in the rationalizing profile and, in turn, efficiency rules out the choice of the last element in the chain in the presence of the first element. The following characterization follows from a result due to Bossert and Sprumont (2003).

Theorem 9 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X} \text{ and } y \in S\}$. C satisfies any of* **E-R-T**, **E-∅-T**, **E-R-Q**, **E-∅-Q** *if and only if C satisfies extended reference-dependent congruence.*

Proof. By Theorem 8, it is sufficient to establish the equivalence of **E-∅-T** and extended reference-dependent congruence.

Suppose first that $(R_1, \ldots, R_n)$ is an E-rationalization of C such that the R_i are transitive. By way of contradiction, suppose there exist $(S, z) \in \Sigma$ and $x, y \in X$ such that $(x, y) \in tc(R_C)$, $x \in S$, $x \neq y$ and $y \in C(S, z)$. Thus, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. By Theorem 6, $(x^{k-1}, x^k) \in R_i$ for all $i \in \{1, \ldots, n\}$. Because the R_i are transitive, it follows that $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$ which, together with the assumptions $x \in S$, $x \neq y$ and $y \in C(S, z)$, contradicts (3).

Now suppose C satisfies extended reference-dependent congruence. Let

$$R_i = tc(R_C)$$

for all $i \in \{1, \ldots, n\}$. Clearly, the R_i are transitive. We complete the proof by establishing that $(R_1, \ldots, R_n)$ is an E-rationalization of C.

The first step in accomplishing this task is to show that $R_i = tc(R_C)$ is antisymmetric. Suppose, by way of contradiction, that there exist two distinct alternatives $x, y \in X$ such that $(x, y) \in tc(R_C)$ and $(y, x) \in tc(R_C)$. Hence, there exist $K, L \in \mathbb{N}$, $x^0, \ldots, x^K \in X$ and $y^0, \ldots, y^L \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$, $x^K = y = y^0$, $(y^{\ell-1}, y^\ell) \in R_C$ for all

$\ell \in \{1, \ldots, L\}$ and $y^L = x$. Without loss of generality, suppose $y^{L-1} \neq x$ (if not, replace y^{L-1} with the highest-numbered y^ℓ that is different from x; this is always possible because $x \neq y$). By definition, we have $(x, y^{L-1}) \in tc(R_C)$. Because $(y^{L-1}, y^L) = (y^{L-1}, x) \in R_C$, there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y^{L-1} \in C(S, x)$. But this contradicts extended reference-dependent congruence because $x \in S$ and $x \neq y^{L-1}$.

Finally, we prove that (1) and (3) are satisfied. (1) follows immediately by definition of R_C and the fact that $R_C \subseteq tc(R_C)$. Now suppose, by way of contradiction, that (3) is violated. Then there exist $(S, y) \in \Sigma$, $x \in C(S, y)$ and $z \in S \setminus \{x\}$ such that $(z, x) \in tc(R_C)$, contradicting extended reference-dependent congruence. ■

Now we turn to a characterization of the axioms in the third box. The conjunction of two axioms turns out to be equivalent to these notions of E-rationalizability. The first property is reference-dependent congruence as defined earlier in the context of single-agent non-deteriorating choice. This property is needed in order to rule out strict-preference cycles resulting from the non-deterioration requirement. However, a second property is needed in order to ensure that efficiency can be satisfied. This second axiom is the weakening of extended reference-dependent congruence that is obtained if the transitive closure of R_C is replaced with R_C itself.

Weak extended reference-dependent congruence. For all $(S, z) \in \Sigma$ and for all $x, y \in X$,

$$[(x, y) \in R_C \text{ and } x \in S \text{ and } x \neq y] \Rightarrow y \notin C(S, z).$$

Reference-dependent congruence and weak extended reference-dependent congruence are independent. To see that reference-dependent congruence does not imply weak extended reference-dependent congruence, suppose $X = \{x, y, z\}$, $\Sigma = \{(\{x, y\}, y), (X, y)\}$, $C(\{x, y\}, y) = \{x\}$ and $C(X, y) = \{y\}$. This reference-dependent choice function satisfies reference-dependent congruence, as is straightforward to verify. Because $x \in C(\{x, y\}, y)$, it follows that $(x, y) \in R_C$ and, together with $x \in X \setminus \{y\}$ and $y \in C(X, y)$, we obtain a violation of weak extended reference-dependent congruence. Now let $X = \{x, y, z\}$, $\Sigma = \{(\{x, y\}, y), (\{x, z\}, x), (\{y, z\}, z)\}$, $C(\{x, y\}, y) = \{x\}$, $C(\{x, z\}, x) = \{z\}$ and $C(\{y, z\}, z) = \{y\}$. This reference-dependent choice function satisfies weak extended reference-dependent congruence. Because $(x, z) \in tc(R_C)$, $x \neq z$ and $y \in C(\{x, z\}, x)$, reference-dependent congruence is violated.

We now obtain the following characterization result.

Theorem 10 *Suppose $C: \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X}$ and $y \in S\}$. C satisfies any of* **E-R-S**, **E-∅-S**, **E-R-A**, **E-∅-A** *if and only if C satisfies reference-dependent congruence and weak extended reference-dependent congruence.*

Proof. By Theorem 8, it is sufficient to establish the equivalence of **E-∅-A** and the conjunction of reference-dependent congruence and weak extended reference-dependent congruence.

Suppose first that $(R_1, \ldots, R_n)$ is an E-rationalization of C such that the R_i are acyclical.

To show that reference-dependent congruence is satisfied, suppose, by way of contradiction, that there exist $S \in \mathcal{X}$ and $x, y \in X$ such that $(x, y) \in tc(R_C)$, $(S, x) \in \Sigma$, $x \neq y$ and $y \in C(S, x)$. Thus, there exist $K \in \mathbb{N}$ and $x^0, \ldots, x^K \in X$ such that $x = x^0$, $(x^{k-1}, x^k) \in R_C$ for all $k \in \{1, \ldots, K\}$ and $x^K = y$. By Theorem 6, $(x^{k-1}, x^k) \in R_i$ for all $i \in \{1, \ldots, n\}$. Because $x \neq y$, we can without loss of generality assume that the x^k are pairwise distinct. Thus, the antisymmetry of the R_i implies $(x^{k-1}, x^k) \in P(R_i)$ for all $i \in \{1, \ldots, n\}$. Furthermore, $y \in C(S, x)$ implies $(y, x) \in R_C$ and, invoking Theorem 6 and antisymmetry again, it follows that $(y, x) = (x^K, x^0) \in P(R_i)$ for all $i \in \{1, \ldots, n\}$. But this contradicts the acyclicity of the relations R_i.

To establish weak extended reference-dependent congruence, suppose, by way of contradiction, that there exist $(S, z) \in \Sigma$ and $x, y \in X$ such that $(x, y) \in R_C$, $x \in S$, $x \neq y$ and $y \in C(S, z)$. Theorem 6 implies $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$. Because $y \in C(S, z)$ and $x \in S \setminus \{y\}$, this contradicts (3).

We now prove the reverse implication. Suppose C satisfies reference-dependent congruence and weak extended reference-dependent congruence. We complete the proof by establishing that R_C is acyclical and that the profile $(R_C, \ldots, R_C)$ is an E-rationalization of C.

To establish the acyclicity of R_C, suppose $(x, y) \in tc(R_C)$. By way of contradiction, suppose $(y, x) \in P(R_C)$. Clearly, this implies $x \neq y$. Furthermore, by definition, there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y \in C(S, x)$. Because $x \neq y$, this contradicts reference-dependent congruence.

Next, we show that R_C is antisymmetric. Suppose, by way of contradiction, that there exist two distinct alternatives $x, y \in X$ such that $(x, y) \in R_C$ and $(y, x) \in R_C$. This implies that there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y \in C(S, x)$. Together with $(x, y) \in R_C$ and $x \neq y$, this contradicts reference-dependent congruence.

Finally, we prove that (1) and (3) are satisfied. (1) follows immediately by definition of R_C. Now suppose, by way of contradiction, that there exist $(S, y) \in \Sigma$, $x \in C(S, y)$ and $z \in S \setminus \{x\}$ such that $(z, x) \in R_C$. This is an immediate contradiction to weak extended reference-dependent congruence and, thus, (3) is satisfied as well. ■

The weakest version of E-rationalizability is characterized by weak extended reference-dependent congruence alone. Because no coherence property is imposed in these notions of E-rationalizability, we do not require any restrictions concerning the transitive closure of R_C. Thus, we obtain the following result.

Theorem 11 *Suppose $C\colon \Sigma \to \mathcal{X}$ is a reference-dependent choice function with an arbitrary non-empty domain $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X}$ and $y \in S\}$. C satisfies any of* **E-RC-∅**, **E-R-∅**, **E-C-∅**, **E-∅-∅** *if and only if C satisfies weak extended reference-dependent congruence.*

Proof. By Theorem 8, it is sufficient to establish the equivalence of **E-∅-∅** and weak extended reference-dependent congruence.

Suppose first that $(R_1, \ldots, R_n)$ is an E-rationalization of C. To show that weak extended reference-dependent congruence is satisfied, suppose, by way of contradiction, that there exist $(S, z) \in \Sigma$ and $x, y \in X$ such that $(x, y) \in R_C$, $x \in S$, $x \neq y$ and $y \in C(S, z)$. Theorem 6 implies $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$. Because $y \in C(S, z)$ and $x \in S \setminus \{y\}$, this contradicts (3).

Now suppose C satisfies weak extended reference-dependent congruence. We complete the proof by establishing that the profile $(R_C, \ldots, R_C)$ is an E-rationalization of C.

To show that R_C is antisymmetric, suppose, by way of contradiction, that there exist two distinct alternatives $x, y \in X$ such that $(x, y) \in R_C$ and $(y, x) \in R_C$. This implies that there exists $S \in \mathcal{X}$ such that $(S, x) \in \Sigma$ and $y \in C(S, x)$. Together with $(x, y) \in R_C$ and $x \neq y$, this contradicts weak extended reference-dependent congruence.

That (1) is satisfied follows immediately by definition of R_C. Now suppose, by way of contradiction, that there exist $(S, y) \in \Sigma$, $x \in C(S, y)$ and $z \in S \setminus \{x\}$ such that $(z, x) \in R_C$. This is an immediate contradiction to weak extended reference-dependent congruence and, thus, (3) is satisfied as well. ■

As mentioned earlier, the requirement **E-RC-T** and its equivalents are more complex than the remaining notions of E-rationalizability. The reason why it is difficult to obtain a characterization result on arbitrary domains for an arbitrary set of agents and an arbitrary universal set of alternatives is that it is not possible to formulate a condition that applies to *all* combinations of the number n of agents and the cardinality $|X|$ of the set of alternatives. The cases where a general characterization can be formulated are those in which **E-RC-T** is equivalent to **E-∅-T** and, thus, Theorem 9 applies. For all other combinations of n and $|X|$, it is not possible to obtain necessary and sufficient conditions that do not depend on n, and the formal problem that results is closely related to the problem of determining the *dimension* of a quasi-ordering; see, for instance, Dushnik and Miller (1941).

Clearly, extended reference-dependent congruence is necessary for **E-RC-T** because, as shown in Theorem 9, the axiom is necessary for **E-∅-T** which obviously is implied by **E-RC-T**. The following theorem, due to Bossert and Sprumont (2003), identifies the combinations of n and $|X|$ for which extended reference-dependent congruence is also sufficient for **E-RC-T**.

Theorem 12 *Suppose* n *and* X *are such that* $|X| < 2(n + 1)$ *and* $C : \Sigma \to \mathcal{X}$ *is a reference-dependent choice function with an arbitrary non-empty domain* $\Sigma \subseteq \{(S, y) \mid S \in \mathcal{X} \text{ and } y \in S\}$. *C satisfies any of* **E-RC-T**, **E-C-T**, **E-RC-S**, **E-C-S**, **E-RC-Q**, **E-C-Q**, **E-RC-A**, **E-C-A** *if and only if C satisfies extended reference-dependent congruence.*

Proof. By Theorem 8, it is sufficient to restrict attention to **E-RC-T**.

That **E-RC-T** implies extended reference-dependent congruence follows immediately from Theorem 9 and the observation that **E-RC-T** implies **E-∅-T**.

Now suppose $|X| < 2(n+1)$ and C satisfies extended reference-dependent congruence. As in the proof of Theorem 9, it follows that $tc(R_C) \cup \Delta$ is reflexive and transitive, and the profile $(tc(R_C) \cup \Delta, \ldots, tc(R_C) \cup \Delta)$ is an E-rationalization of C.

The following definitions will be used in the remainder of the proof. The *dimension* of a quasi-ordering R on X is the smallest positive integer r with the property that there exist r orderings $R_1, \ldots, R_r$ whose intersection is R. For a real number α, the largest integer less than or equal to α is denoted by $[\alpha]$.

Next, we show that the dimension of $tc(R_C) \cup \Delta$ does not exceed n. If $|X| \leq 3$, the dimension of $tc(R_C) \cup \Delta$ is less than or equal to two which, in turn, is less than or equal to n. If $|X| \geq 4$, Hiraguchi's inequality (see Hiraguchi, 1955) implies that the dimension of $tc(R_C) \cup \Delta$ is less than or equal to $[|X|/2]$. Because $|X| < 2(n+1)$ implies $[|X|/2] \leq n$, it follows again that the dimension of $tc(R_C) \cup \Delta$ is less than or equal to n.

Thus, there exist antisymmetric orderings $R_1, \ldots, R_n$ (not necessarily distinct) on X whose intersection is $tc(R_C) \cup \Delta$. It is now straightforward to verify that the profile $(R_1, \ldots, R_n)$ is an E-rationalization of C. ■

As an immediate corollary of this result and Theorem 9, it follows that **E-∅-T** and **E-RC-T** are equivalent whenever $|X| < 2(n+1)$.

The above theorem is tight in the sense that the assumption $|X| < 2(n+1)$ cannot be weakened: whenever $|X| \geq 2(n+1)$, it is possible to find a reference-dependent choice function C satisfying extended reference-dependent congruence and violating **E-RC-T**. This observation is also due to Bossert and Sprumont (2003) and it can be proven by employing the example of Part **(e)** in the proof of Theorem 8.

We conclude this section with an analysis of the consequences of a specific domain assumption. In particular, we employ what Bossert and Sprumont (2003) refer to as *universal set domains.* These domains are such that, for every reference-dependent choice problem, the entire set X is feasible and only the reference alternative is allowed to vary from one reference-dependent choice problem to another. Thus, Σ is a universal set domain if and only if $S = X$ for all $(S, y) \in \Sigma$.

The assumption that Σ is a universal set domain has remarkably strong consequences. Under this assumption, *all* notions of E-rationalizability coincide.

Theorem 13 *Suppose* $\beta, \beta' \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \emptyset\}$, $\gamma, \gamma' \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \emptyset\}$ *and* $C\colon \Sigma \to \mathcal{X}$ *is a reference-dependent choice function with a non-empty universal set domain* $\Sigma \subseteq \{(X, y) \mid y \in X\}$. C *satisfies* **E-**β**-**γ *if and only if* C *satisfies* **E-**β'**-**γ'.

Proof. Suppose C is a reference-dependent choice function defined on a universal set domain Σ. Given that all equivalences of Theorem 8 remain valid on universal set domains, it is sufficient to prove that **E-∅-∅** implies **E-C-T**. Let $(R_1, \ldots, R_n)$ be an E-rationalization of C. Define the relation R^* on X by

$$R^* = \{(x, y) \mid x \in C(\Sigma) \text{ and } y \in X \setminus C(\Sigma)\}.$$

To see that R^* is transitive, suppose $(x, y) \in R^*$ and $(y, z) \in R^*$. By definition of R^*, $(x, y) \in R^*$ implies $y \notin C(\Sigma)$ and $(y, z) \in R^*$ implies $y \in C(\Sigma)$, which is impossible. Therefore, transitivity is vacuously satisfied. Analogously, the definition of R^* immediately implies that it is impossible to have both $(x, y) \in R^*$ and $(y, x) \in R^*$ and, therefore, R^* is antisymmetric.

Next, we prove that $R_C \subseteq R^*$. Suppose $(x, y) \in R_C$. The definition of R_C implies $(X, y) \in \Sigma$ and $x \in C(X, y)$. Thus, $x \in C(\Sigma)$. Because $(R_1, \ldots, R_n)$ is an E-rationalization of C, it follows that $(x, y) \in R_i$ for all $i \in \{1, \ldots, n\}$. If $y \in C(\Sigma)$, there exists $z \in X$ such that $(X, z) \in \Sigma$ and $y \in C(X, z)$. Because $x \in X \setminus \{y\}$, this contradicts the efficiency property (3) implied by **E-∅-∅**. Thus, $y \in X \setminus C(\Sigma)$ and, by definition, $(x, y) \in R^*$.

Let R^{**} be an arbitrary complete, transitive and antisymmetric relation defined on $X \setminus C(\Sigma)$, and let $R^0 = R^* \cup R^{**}$. Clearly, R^0 is a transitive and antisymmetric relation such that $R_C \subseteq R^* \subseteq R^0$. According to the relation R^0, the non-comparable pairs (x, y) are all such that both x and y are in $C(\Sigma)$.

Now let R_1^0 be an arbitrary complete, transitive and antisymmetric relation defined on $C(\Sigma)$, and let $R_2^0 = \ldots = R_n^0$ be the inverse of R_1^0, that is,

$$R_i^0 = \{(x, y) \mid (y, x) \in R_1^0\}$$

for all $i \in \{2, \ldots, n\}$. Define

$$R_i' = R^0 \cup R_i^0$$

for all $i \in \{1, \ldots, n\}$. It is straightforward to verify that the R_i' are complete, transitive and antisymmetric. To conclude the proof that $(R_1', \ldots, R_n')$ is an E-rationalization of C, note first that (1) follows immediately from the observation that $R_C \subseteq R_i'$ for all $i \in \{1, \ldots, n\}$. To establish (3), suppose $x \in C(\Sigma)$. This implies $(x, y) \in R^*$ and thus $(x, y) \in R_i'$ for all $y \in X \setminus C(\Sigma)$ and for all $i \in \{1, \ldots, n\}$ and, by antisymmetry, no $y \in X \setminus C(\Sigma)$ can be such that $(y, x) \in R_i'$ for all $i \in \{1, \ldots, n\}$. Furthermore, we have

$$(x, y) \in R_1' \text{ or } (x, y) \in R_2'$$

for all $y \in C(\Sigma) \setminus \{x\}$ by definition and, thus, $(y, x) \in R_i'$ for all $i \in \{1, \ldots, n\}$ is not possible for any $y \in C(\Sigma) \setminus \{x\}$ either. ■

Our final result provides a characterization of E-rationalizability if Σ is a universal set domain. In this case, the relevant congruence axiom is defined as follows.

Universal reference-dependent congruence. For all $(X, z) \in \Sigma$ and for all $x, y \in X$,

$$[(x, y) \in R_C \text{ and } x \neq y] \Rightarrow y \notin C(X, z).$$

This axiom characterizes the single version of E-rationalizability that exists in the presence of a universal set domain.

Theorem 14 *Suppose $C: \Sigma \to \mathcal{X}$ is a reference-dependent choice function with a non-empty universal set domain $\Sigma \subseteq \{(X, y) \mid y \in X\}$. C satisfies* **E**-β-γ *for any $\beta \in \{\mathbf{RC}, \mathbf{R}, \mathbf{C}, \emptyset\}$ and any $\gamma \in \{\mathbf{T}, \mathbf{Q}, \mathbf{S}, \mathbf{A}, \emptyset\}$ if and only if C satisfies universal reference-dependent congruence.*

Proof. By Theorem 13, it is sufficient to establish the equivalence of **E**-$\emptyset$-$\emptyset$ and universal reference-dependent congruence.

Suppose first that $(R_1, \ldots, R_n)$ is an E-rationalization of C and, by way of contradiction, suppose there exist $(X, z) \in \Sigma$ and $x, y \in X$ such that $(x, y) \in R_C$, $x \neq y$ and $y \in C(X, z)$. By Theorem 6, $(x, y) \in R_i$ and, by antisymmetry, $(x, y) \in P(R_i)$ for all $i \in \{1, \ldots, n\}$. Because $y \in C(X, z)$ and $x \in X \setminus \{y\}$, this contradicts efficiency.

Now suppose C satisfies universal reference-dependent congruence. To complete the proof, we show that $(R_C, \ldots, R_C)$ is an E-rationalization of C.

First, we establish the antisymmetry of R_C. Suppose $(x, y) \in R_C$ for two distinct alternatives $x, y \in X$. If $(X, x) \notin \Sigma$, it is immediate that $(y, x) \notin R_C$. If $(X, x) \in \Sigma$, universal reference-dependent congruence implies $y \notin C(X, x)$.

That (1) is satisfied follows immediately from the definition of R_C. Finally, suppose (3) is violated. Then there exist $(X, y) \in \Sigma$, $x \in C(X, y)$ and $z \in X \setminus \{x\}$ such that $(z, x) \in R_C$. This contradicts universal reference-dependent congruence and, thus, (3) is satisfied. ■

3. Concluding Remarks

A notion of non-deteriorating choice, which is proposed as an alternative to the traditional notion of greatest-element rational choice and maximal-element rational choice, is characterized on general domains and without full transitivity of rationalizing relations. The logical structure of our analyses as well as the characterizing axioms, both in the context of single-agent choice and multi-agent choice, are made parallel to the traditional rational choice theory as much as possible with the purpose of facilitating comparisons with the traditional theory. Except for the multi-agent non-deteriorating and efficient choice on general domains and with full transitivity of rationalizing relations, we have provided complete characterizations of all the cases of interest, thus narrowing down the class of problems to be explored further in the future. Among the problems we may pose in the context of multi-agent choice is the characterization of *core rationalizability* introduced by Bossert and Sprumont (2002) in the context of two-person exchange economies, which is a sophistication of the notion of multi-agent non-deteriorating and efficient choice. Since there are already many dishes on the table, we leave this problem for future exploration.

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