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Indifferences and Domain Restrictions*

Abstract: The purpose of this paper is to discuss the extent to which allowing for individuals to be indifferent among alternatives may alter the qualitative results that are obtained in social choice theory when domain restrictions are defined on profiles of linear orders. The general message is that indifferences require attention and careful treatment, because the translation of results from a world without indifferences to another where agents may be indifferent among some alternatives is not always a straightforward exercise. But the warning is not one-directional: sometimes indifferences complicate the statement of results, but preserve their essential message. Sometimes, they help to create domains where some rules work better than in the presence of linear orders. In other cases, however, their presence destroys the positive results that would apply in their absence. I provide examples of these three situations.

0. Introduction

Most of the positive results in social choice theory apply to families of preference profiles that satisfy some special conditions. Maybe the best known among these conditions is Duncan Black's notion of single peakedness[1], a property that guarantees that simple majority voting and other voting procedures, when defined on profiles of preferences that meet this requirement, may satisfy many desiderata that are otherwise hard to comply with. But there are many other relevant conditions that one may impose on preference profiles and which prove helpful in obtaining interesting results. For a very thorough survey on the subject, including discoveries of his own, see Gaertner (2001).

Conditions that are satisfied by some preference profiles and not by others are said to define a domain restriction, because typically the subset of profiles meeting them is taken as the domain where a restricted social choice rule is defined, and required to perform adequately. Of course, out of the infinite variety of conceivable conditions inducing domain restrictions, social choice theorists have concentrated in those satisfying two natural requirements. One is that the

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[1] Gaertner 2005 has been able to trace the origins of this notion to Pufendorf, but it was certainly Black who introduced it into modern economics and political science.

condition should have some natural interpretation, suggesting that those profiles satisfying it are those that one would find in some voting contexts of economic or political relevance. The other is that the domain, in addition to being relevant, must allow for some attractive voting rules to be defined on it. Single peakedness meets both criteria. It arises as a natural restriction in political and economic contexts, and it provides a domain restriction within which the simple majority rule satisfies all kinds of good properties.

In cases where agents face a finite set of alternatives, it is not unnatural to assume that they have strict preferences on each pair of alternatives, and are never indifferent between any two. Formally, this assumption amounts to let the agents' preferences over alternatives be expressed as a linear order. The absence of indifferences tends to simplify the expression of conditions over preference profiles defining domain restrictions. It also simplifies the proofs of results based on restricting the domains of social choice rules. In some cases, extending the results that one can obtain under the assumption of linear preferences to a wider class of preferences, where agents are allowed to express indifferences, is a rather straightforward exercise[2]. But, as we shall see, sometimes this extension is nontrivial, or even impossible.

The purpose of this essay is to discuss the extent to which allowing for individuals to be indifferent among alternatives may alter the qualitative results that are obtained in social choice theory when domain restrictions are defined on profiles of linear orders[3]. The general message is that indifferences require attention and careful treatment, because the translation of results from a world without indifferences to another where agents may be indifferent among some alternatives is not always a straightforward exercise. But the warning is not one-directional: sometimes indifferences complicate the statement of results, but preserve their essential message. Sometimes, they help to create domains where some rules work better than in the presence of linear orders. In other cases, however, their presence destroys the positive results that would apply in their absence. I will provide examples of these three situations, in order to make my point. Of course, these remarks assume that the reader has sympathy for the idea that, in many contexts, it is natural to assume that agents' preferences will be indifferent among some alternatives.

In Section 1 I formulate the condition of single peakedness and summarize some results that hold under the restricted domains that this condition helps to define.

In Section 2, I define a first extension of single peakedness and introduce the notion of single plateaud preference profiles, which allows individuals to be indifferent among several best alternatives. Under the resulting extended domain

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[2] This is the case, for example, for standard proofs of the Gibbard-Satterthwaite theorem. See Gibbard 1973 or Schmeidler and Sonnenschein 1978.

[3] Notice that in the formal part of the paper linear preferences are not assumed, because the standard notion of single peakedness, which will be my starting point, only allows for some very limited indifference classes. I refer to linear preferences in this introduction to stress the point that indifferences cannot be assumed universally without consequences, and that the type of indifferences that are allowed in each domain can make a difference.

I state results which are more complicated than but qualitatively similar to those in the previous section.

Section 3 discusses another extension of single peakedness that allows for indifferences among the agents' worse alternatives. This case arises naturally when individuals can refer to outside options. In contrast with the preceding case, allowing for indifferences of this sort completely destroys the results that would be obtained in their absence.

In Section 4 I no longer take single peakedness as a reference. Instead, I exhibit a family of new domain restrictions based on the systematic appearance of indifferent alternatives in the preferences of voters. And I show that, in this case, indifferences may actually help in obtaining positive results.

I hope that the examples provided in these three sections give support to my simple messages to the reader. One is that the extent to which voters are assumed to be possibly indifferent among alternatives is relevant when considering the implications of any domain restriction. The other is that one can neither credit, nor blame indifferences in general as being responsible for the improvement or the deterioration of results that would hold in their absence. In order to attribute such responsibilities, one must be precise about the position of permissible indifferences within the preference relation of each individual, and eventually across the preferences of different individuals.

1. Some Simple and Powerful Results Based on Single Peaked Preference Profiles

The purpose of this section is to formulate some well-known results that will be compared later with their eventual counterparts in more complex contexts.

Let A be a set of alternatives (finite or infinite), N be a (finite) set of agents or voters. Consider voter's preferences over the alternatives to be complete, reflexive and transitive binary relations on A. The set of all preferences on A will be denoted by $\Re$.

Denote the preferences of $i \in N$ by $\succcurlyeq_i \in \Re$. The strict part $\succ_i$of $\succcurlyeq_i$is defined so that, for any $x, y \in A, x \succ_i y \iff [x \succcurlyeq_i y$ and not $y \succcurlyeq_i x]$. The symmetric part $\sim_i$ of $\succcurlyeq_i$ is defined so that $x \sim_i y \iff [x \succcurlyeq_i y$ and $y \succcurlyeq_i x]$.

Denote by $t(\succcurlyeq_i)$ the set of maximal elements of A according to $\succcurlyeq_i$ and call this set the top of $\succcurlyeq_i$. When this top is a singleton, call its unique element the peak of $\succcurlyeq_i$ (or the peak of i), and denote it by $p(\succcurlyeq_i)$.

Where n is the cardinality of N, preference profiles are elements of $\Re^n$, denoted by $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n), \succcurlyeq' = (\succcurlyeq'_1, ..., \succcurlyeq'_n)$, etc. Given a preference profile $\succcurlyeq \in \Re^n$, an agent i and a preference $\succcurlyeq_i \in \Re$, denote by $\left(\succcurlyeq_{-i}, \succcurlyeq_i^{'}\right)$ the preference profile obtained from $\succcurlyeq$ after substituting $\succcurlyeq_i$ by $\succcurlyeq_i^{'}$.

I begin with the standard definition of single peaked preference profiles, introduced by Duncan Black (1948).

Definition 1 *A preference profile $\succcurlyeq$ is single peaked relative to a linear order $>$ of the set of alternatives iff*

(1) each of the voters' preferences has a unique maximal element $p(\succcurlyeq_i)$ *and*
(2) for all $i \in N$, *for all* $p(\succcurlyeq_i)$ *and for all* $y, z \in A$, $[p(\succcurlyeq_i) > y > z$ *or* $z > y > p(\succcurlyeq_i)] \iff y \succ_i z$.

Notice that single peakedness requires each agent to have a unique maximal element. It must also be true for any agent that any alternative z to the right (left) of its peak is preferred to any other that is further to the right (left) of it. In particular, this implies that no agent is indifferent between two alternatives on the same side of the peak. Moreover, indifference classes may consist of at most two alternatives, one on each side of the peak. This is a strong restriction on the possibility of agents to show indifference among alternatives, and this restriction is there for a purpose: as we shall see later, lifting it is a delicate matter.

I will denote by Λ the subset of $\Re^n$ which consists of all preference profiles which are single peaked. Say that a preference profile is single peaked if it meets the above conditions for some $>$. In this sense, single peakedness is not directly a property of individual preferences. Yet, once we fix a given $>$, we can focus attention on the subset of individual preferences satisfying conditions (1) and (2) of Definition 1, relative to this order. I denote this subset of $\Re$ by $I\Lambda(>)$.

Also notice that, if we denote by $\Lambda(>)$ the subset of preference profiles which are single peaked relative to an order $>$, we have that $\Lambda(>) = \prod_{i=1}^{n} I\Lambda(>)$. That is, $\Lambda(>)$ is a cartesian product, in contrast with the fact that the more general set Λ is not. This qualification is relevant since, as we shall see, some well known possibility results in social choice are obtained under the assumption that the set of profiles on which to define different rules satisfy single peakedness. Yet, we shall also see that the domain involved in such results is sometimes Λ, sometimes $\Lambda(>)$.

Indeed, if one restricts attention to social choice rules defined on single peaked preference profiles (in one version or the other), it is possible to circumvent some of the major difficulties that arise when trying to design satisfactory rules on universal domains. We will refer specifically to two essential difficulties.

The first one is the appearance of cycles in the social preferences generated by many of the rules that aggregate n-tuples of preferences into a social binary relation. By contrast, we know that the simple majority rule will avoid the problem of cycles when restricted to operate on single peaked preference profiles. We also know of circumstances when full transitivity of the rule can be guaranteed, and also about the connection between the social preferences and those of the median voter.

The following definitions and results summarize part of this knowledge[4].

[4] For more sophisticated versions, see Fishburn 1973, chapters 8–11.

1.1 The Existence and Location of Condorcet Alternatives and of Social Orders

I begin with some definitions.

For any preference profile $\succcurlyeq = (\succcurlyeq_1, ..., \succcurlyeq_n)$, the majority relation induced by the simple majority rule $M(\succcurlyeq_1, ..., \succcurlyeq_n)$ is defined so that for all

$x, y \in A$, $xM(\succcurlyeq_1, ..., \succcurlyeq_n)y \iff \# \{i; x \succcurlyeq_i y\} \geq \# \{j; y \succcurlyeq_j x\}$.

The Condorcet winner(s) at preference profile $\succcurlyeq$ is (are) the maximal element(s) on A according to the social relation induced by the simple majority rule.

For any preference profile $\succcurlyeq \in N(>)$ which is single peaked relative to an order $>$, let $P_{\succcurlyeq}$ be the set of all the alternatives which are the peak for some agent in profile $\succcurlyeq$. Define now the high median peak $p^+(\succcurlyeq)$ for this profile as the smallest alternative (according to $>$) such that

$$\#\{p(\succcurlyeq_i) \in P_{\succcurlyeq}; p(\succcurlyeq_i) > p^+(\succcurlyeq)\} \quad \begin{array}{l} < \frac{n}{2} \text{ (if } n \text{ is even)} \\ < \frac{n+1}{2} \text{ (if } n \text{ is odd)} \end{array}$$

Similarly, the low median peak $p^-(\succcurlyeq)$ is the largest alternative (according to $>$) such that

$$\#\{p(\succcurlyeq_i) \in P_{\succcurlyeq}; p^-(\succcurlyeq) > p(\succcurlyeq_i)\} \quad \begin{array}{l} < \frac{n}{2} \text{ (if } n \text{ is even)} \\ < \frac{n+1}{2} \text{ (if } n \text{ is odd)} \end{array}$$

If $p^-(\succcurlyeq) = p^+(\succcurlyeq)$, then this unique alternative is called the median peak.

Notice that, if n is odd, there is always a median peak.

Agents whose peak is the median peak, are called the median voters.

We can now state a number of known results regarding the majority relation over single peaked profiles.

Theorem 1 *The majority relation induced by the simple majority rule under any single peaked preference profile always satisfies quasitransitivity. Hence, Condorcet winners always exist at these profiles, and they are the alternatives in the interval*

$[p^-(\succcurlyeq), p^+(\succcurlyeq)]$[5]. *When the number of voters is odd, the majority relation is transitive, and there is a unique Condorcet winner, which is the median alternative.*

1.2 Strategy-proofness

Another important and hard to meet condition in social choice is that of strategy proofness. In its standard version, this condition refers to procedures that Gibbard called voting schemes selecting one and only one alternative for each preference profile.

[5] When A is finite then this set consists of at most two elements.

Again, I first provide definitions and continue with an important and well known result.

Let $D = D^1 \times D^2 \times, ..., \times D^N$ be any set of preference profiles that can be expressed as the cartesian product of families of individual preferences.

A social choice function on D is a function $f : D \to A$.

A social choice function on D is strategy proof iff, for all preference profiles $\succcurlyeq \in D \equiv \prod_{j=1}^{n} D^j$, all agents i and all preferences $\succcurlyeq_i' \in D^i$, $f(\succcurlyeq) \succcurlyeq_i f(\succcurlyeq_{-i}, \succcurlyeq_i')$.

A social choice function is dictatorial on D if there exists an agent $i \in N$ such that, for any preference profile in D, $f(\succcurlyeq_1, ..., \succcurlyeq_n)$ is a maximal element of $\succcurlyeq_i$ on the range r_f of f.

A celebrated theorem due to Gibbard (1973) and Satterthwaite (1975) establishes that any social choice function on the universal domain of preferences whose range contains more than two alternatives is either dictatorial or manipulable.

However, Moulin (1980) proved that it is possible to define non-dictatorial strategy proof social choice functions in the Cartesian domains of all preference profiles that are single peaked relative to a given order $>$. In fact, he provided a characterization of such rules. For ease of exposition, we identify here (without loss of generality) the set of alternatives A with a closed interval $[a, b]$ of the real line, if A is a continuum of alternatives, or with an integer interval $[a, a+1, ..., b]$, if it is a finite set. In that case, $>$ is the natural order of numbers.

Construction. For each coalition $S \in 2^N \backslash \emptyset$, fix an alternative a_s. Define a social choice function in such a way that, for each preference profile $(\succcurlyeq_1, ..., \succcurlyeq_n)$,

$$f(\succcurlyeq_1, ..., \succcurlyeq_n) = \min_{S \subset N} \left[\max_{i \in S} (a_s, p(\succcurlyeq_i)) \right]$$

The functions so defined will be called generalized median voter schemes.

Notice that, if preferences are defined on $[a, b]$, the generalized median voter scheme is efficient iff $a_\phi = b$ and $a_N = a$.

Theorem 2 *(Moulin, 1980a) A social choice function on profiles of single peaked preferences over a totally ordered set is strategy proof if and only if it is a generalized median voter scheme.*[6]

Single-peakedness is, therefore, a domain restriction that allows for very nice and positive results. However, I have already noted that it establishes a very strong requirement over individual preferences. Any alternative can only be indifferent with at most another one. Since this is not always natural, we shall explore the consequences of admitting further indifferences on the part of the individuals.

[6] Alternative characterization is provided in Barberà, Gul and Stachetti 1993. See also Barberà 2001, and forthcoming.

Actually, Moulin's initial statement was proven for those voting schemes which satisfy the "tops only" property. That is, for those who are defined as functions of the peak of each of the agent's preferences. Later on, Barberà and Jackson 1994 as well as Sprumont 1990 did prove that this property was unnecessary in the characterisation, since it is implied by strategy proofness and ontoness in Moulin's context.

2. From Single Peaked to Single Plateaud Preferences

One simple way to admit that agents can be indifferent among more alternatives than in the case of single peakedness arises by allowing that each voter may consider that more than one alternative is best. That is, each agent may be allowed to be indifferent between a number of alternatives, all of which are consecutive with respect to a given order and are then preferred to the rest. When, in addition, a natural extension of the single peakedness condition does hold, we find the rather well known case of single plateaud preferences.

Definition 2 *A preference profile is single plateaud relative to an order* $>$ *of the set of alternatives iff*

(1) each of the voters' top $t(\succcurlyeq_i)$ *is an interval (i. e., is such that, for all* $x, y \in t(\succcurlyeq_i)$*, and all* z*,* $x > z > y$ *implies that* $z \in t(\succcurlyeq_i)$*).*

(2) for all $i \in N$*, for all* $x \in t(\succcurlyeq_i)$*, all* $y, z \notin t(\succcurlyeq_i)$*,* $[x > y > z$ *or* $z > y > x] \iff y \succ_i z$*.*

I will denote by $t^+(\succcurlyeq_i)$ (resp. $t^-(\succcurlyeq_i)$) the alternative in $t(\succcurlyeq_i)$ which is largest (resp. smallest) according to $>$.

I will denote by $\bar{\wedge}$ the set of all profiles that are single plateaud and by $\bar{\wedge}(>)$ the set of all those that are single plateaud with respect to a given order. The same remarks regarding the cartesian structure of the latter, and not of the former, do apply.

2.1 Extending Black's Result

In the presence of such a condition, it is not trivial to extend Black's or Moulin's results. However, it is possible to do it, while preserving the flavor of the original results to a large extent.

The clues for an extension of Black's results are brilliantly provided in Fishburn (1973). In this wonderful book, the author proves a general result (Theorem 9.3, 108) which covers many cases and complications. I will adapt it and simplify it here for expository purposes, and also to draw a parallel as exact as possible with the version I gave of Black's results.

For the purpose of the following construction, the two elements $t^+(\succcurlyeq_i)$ and $t^-(\succcurlyeq_i)$ are treated as two different objects, even if they both correspond to the same alternative. Likewise, when two agents contribute the same alternative to the list, this alternative is counted as a different object each time.

Let $T_{\succeq} = \bigcup_{i=1}^{n} \{t^+(\succcurlyeq_i), t^-(\succcurlyeq_i)\}$.

Notice that, under the previous remarks, the set T consists of $2n$ objects $t_1, ..., t_{2n}$.

For any single plateaud preference profile $\succcurlyeq$, relative to an order $>$ the high median top $t^+(\succcurlyeq)$ is the smallest alternative (according to $>$) such that $\#\{t_h \in T_{\succeq}; t_h > t^+(\succcurlyeq)\} < n$.

Similarly, the lower median top $t^{-}(\succcurlyeq)$ is the largest alternative (according to $\succ$) such that $\#\{t_j \in T_{\succeq}; t_j < t^{-}(\succcurlyeq)\} < n$.

If there exists some agent i such that $t^{+}(\succcurlyeq) = t^{+}(\succcurlyeq_i)$ and $t^{-}(\succcurlyeq) = t^{-}(\succcurlyeq_i)$, then we call this agent the median agent at profile $\succcurlyeq$.

I now describe results that parallel Theorem 1 for the case of single plateaud preferences.

Theorem 3 *The social relation induced by the simple majority rule under any single plateaud preference profile always satisfies quasitransitivity. Hence, Condorcet winners always exist at these profiles, and they are the alternatives in the interval* $[t^{-}(\succcurlyeq), t^{+}(\succcurlyeq)]$. *When the number of voters is odd, the majority relation is transitive. Notice that if there is a median agent, then its top coincides with the set of Condorcet winners.*

2.2 Extending Moulin's Result

Turning now to the extension of Moulin's characterization of strategy proof rules to the case of single plateaud preferences, I will state a result due to Berga (1998), which is enough to warn the reader about the many additional subtleties that lurk behind any changes in the domain of definition of such rules.

Intuitively, the result is simple. Based on a simple construction. Choose rules that, for each single plateaud preference in a profile, select one of the maximal alternatives in this plateau. Then, use any of Moulin's strategy proof rules for single plateaud preferences to compute what would be the outcome if agents' top consisted only of the selected peak. In spite of this natural connection with Moulin's original result, the reader will be able to recognize in Berga's careful statement how much one has to watch for in order to accommodate it to the new domain.

First of all, there is a loss. The "tops only" condition must be imposed, since there is no clear parallel to the proof that such a property is implied by strategy proofness, as in the case under single peaked preferences. Then, the family of rules to break ties must be carefully delimited. If too few are allowed, then one cannot reach a characterization. If too many, then some may introduce a possibility of manipulation.

Definition 3 *A social choice function f is plateau-only if for any $\succcurlyeq, \succcurlyeq' \in \overline{\wedge}(>)$ such that for any $i \in N, t(\succcurlyeq_i) = t(\succcurlyeq'_i)$, then $f(\succcurlyeq) = f(\succcurlyeq')$.*

Definition 4 *A social choice function f is uncompromising if the following holds: pick $\succcurlyeq \in \overline{\wedge}(>)$ and set $f(\succcurlyeq) = z$. For any $j \in N$, any $\succcurlyeq'_j \in \overline{\wedge}(>)$, if either (i) $z < t^{-}(\succcurlyeq_j)$ and $z \leq t^{-}(\succcurlyeq'_j)$, or (ii) $z > t^{+}(\succcurlyeq_j)$ and $z \geq t^{+}(\succcurlyeq'_j)$ holds, then $f(\succcurlyeq_{-j}, \succcurlyeq'_j) = f(\succcurlyeq)$.*

Definition 5 *A tie-breaking rule for agent i is a function $h_i : [\overline{\wedge}(>)]^n \to A$ such that for any $\succcurlyeq \in [\overline{\wedge}(>)]^n$ and all $i \in N, h_i(\succcurlyeq) \in t(\succcurlyeq_i)$.*

Definition 6 *A tie-breaking rule for agent i, h_i is plateau-only if for any $\succcurlyeq, \succcurlyeq' \in \overline{\curlywedge}^n$ such that for any $j \in N, t(\succcurlyeq_j) = t(\succcurlyeq'_j)$, then $h_i(\succcurlyeq) = h_i(\succcurlyeq')$.*

Definition 7 *A tie-breaking rule for agent i, h_i is strategy proof if for any $\succcurlyeq \in [\overline{\curlywedge}(>)]^n$, any $j \in N \backslash \{i\}$ and any $\succcurlyeq'_j \in [\overline{\curlywedge}(>)]^n, h_i(\succcurlyeq) \succcurlyeq_j h_i(\succcurlyeq_{-j}, \succcurlyeq'_j)$. Otherwise, h_i is said to be manipulable.*

Notice that since the outcome given by h_i belongs to the plateau of agent i, only agents in $N \backslash \{i\}$ can gain by misrepresenting their preferences.

Let us call $h = (h_i)_{i \in N}$, a list of the tie-breaking rules for each agent, a tie-breaking rule for the society. We say that h is componentwise plateau-only if for any $i \in N$, h_i is plateau-only. Moreover, h is componentwise strategy proof if for any $i \in N$, h_i is strategy proof.

By definition, h selects an n-tuple of points which belong to the n-tuple of plateaux. We emphasize that this selection may depend on the whole profile.

We are now able to define the class of social choice functions that will play a crucial role in our results.

Definition 8 *A social choice function $f : [\overline{\curlywedge}(>)]^n \rightarrow A$ is called tie-breaking minmax social choice function if there exist a tie-breaking rule h and a list of parameters $\{a_s\}_{S \subseteq N}$ in A such that, for any $\succcurlyeq \in [\overline{\curlywedge}(>)]^n$,*

$$f(\succcurlyeq) = \min_{S \subseteq N} \{ \max_{i \in S} \{h_i(\succcurlyeq), a_s\}\}.$$

Note that if the tie-breaking rule h is componentwise plateau-only, then the minmax function is plateau-only.

The following characterization singles out the class of tie-breaking minmax plateau-only social choice functions with componentwise strategy proof tie-breaking rule for the society as the only class of strategy proof plateau-only social choice functions.

Theorem 4 *A plateau-only social choice function $f : [\overline{\curlywedge}(>)]^n \rightarrow A$ is strategy proof if and only if f is a tie-breaking minmax plateau-only social choice function whose associated tie-breaking rule is componentwise strategy proof.*

Berga can also prove another important result, where the tops only requirement is substituted by the condition that the social choice function be uncompromising.

Definition 9 *A tie-breaking rule for agent i, h_i is uncompromising if the following holds: pick any $\succcurlyeq \in [\overline{\curlywedge}(>)]^n$ and set $h_i(\succcurlyeq) = z$. For any $j \in N \backslash \{i\}$, any $\succcurlyeq_j \in \overline{\curlywedge}(>)$, if either $(i) z < t^-(\succcurlyeq_j)$ and $z \leqslant t^-(\succcurlyeq'_j)$, or (ii) $z > t^+(\succcurlyeq_j)$ and $z \geqslant t^+(\succcurlyeq'_j)$ holds, then $h_i(\succcurlyeq_1, ..., \succcurlyeq'_j, ..., \succcurlyeq) = h_i(\succcurlyeq)$.*

We say that a tie-breaking rule for the society h is componentwise uncompromising if for any agent $i \in N$, h_i is uncompromising.

Theorem 5 *A social choice function $f : [\overline{\curlywedge}(>)]^n \rightarrow A$ is strategy proof and uncompromising if and only if f is a tie-breaking minmax social choice function whose associated tie-breaking rule is componentwise both strategy proof and uncompromising .*

Having seen how the passage from single peaked to single plateaud preference profiles complicates results without destroying their basic spirit, I now turn to the description of other natural domains allowing for indifferences which do have more severe consequences.

3. From Single Peakedness to Outside Options

This section is inspired by the work of Cantala (2004). In one of its leading interpretations, single peakedness on a line arises as a natural consequence of a preference for having a public facility as close as possible from home. Distance to the point of consumption of the public facility induces disutility. Under the same interpretation, one may add in the model the possibility of an outside option, an alternative way to satisfy needs at a cost. In the presence of such an outside option, we'll assume that agents will only use the public facility if it is "close enough" to their location. Therefore, it is natural to think that preferences will have the single peaked shape up to certain "outside" limits, determined by the cost to each agent of using the outside option, and then will become "flat". That is: locations of the public facility which are too far from the peak will be indifferent to each other.

I now provide a condition, proposed by Cantala, which formalizes this type of situations. Again, as in Section 2, I assume without loss of generality that A is an interval and that $>$ is the natural order on the reals.

Definition 10 *A preference profile is single peaked with outside options relative to an order* $>$ *on* A *iff, for each agent* i*, there exists an interval* $C(\succcurlyeq_i) \equiv [l(\succcurlyeq_i), U(\succcurlyeq_i)] \subset A$ *such that*

(a) for all $x \in C(\succcurlyeq_i)$ and $y \in A \backslash C(\succcurlyeq_i), x \succ_i y$

(b) for all $z, w \in A \backslash C(\succcurlyeq_i)$, (i.e., such that $z > h(\succcurlyeq_i)$ or $U(\succcurlyeq_i) > z$, and $w > h(\succcurlyeq_i)$ or $U(\succcurlyeq_i) > w$), $z \sim_i w$

(c) for all i, there exists a single peak $p(\succcurlyeq_i)$,and for all $r, s \in C(\succcurlyeq_i)$,

$$\begin{matrix} r > s > p(\succcurlyeq_i) \\ p(\succcurlyeq_i) > s > r \end{matrix} \Longrightarrow s >_i r$$

Let $O(>)$ be the set of preferences on $A \equiv [a, b]$ which are single peaked with an outside option relative to order $>$.

Notice that, the definition is quite similar to that of single peakedness, but now people may have large indifference classes regarding extreme alternatives. This apparently innocuous change has rather dramatic consequences.

3.1 Troubles with Condorcet Winners

In fact, Cantala exhibits three examples, each one of which disrupts, in an increasing sequence of intensity, Black's median voter result.

Indeed, in his first example, all the alternatives which are the peak for some agent qualify as Condorcet winners.

Example 1 *Let* $N = \{1, 2, 3\}$ *be the set of agents. Whenever the interval of consumption of the agents are pairwise disjoint, each peak is a Condorcet winner.*

Second, the median peak may not be a Condorcet winner.

Example 2 *Let* $N = \{1, 2, 3, 4, 5\}$ *be the set of agents. Whenever the interval of acceptable levels of the first four agents, say 1, 2, 3 and 4 from left to right, are disjoint and preferences of the last agent coincide with those of agent 1, then the Condorcet winner is the peak of agent 1 whereas that of agent 2 is the median peak.*

The third example is still more destructive: there may not exist any Condorcet winner when preferences are single peaked with outside options.

Example 3 *Let* $N = \{1, 2, 3, 4, 5\}$ *be the set of agents and consider the preferences as represented in Fig. 1, where the horizontal axis is the reservation utility (the utility generated by the outside option) normalized for all agents. There, for all* $t \leq a$*, alternative* z *beats* t*; for all* $a \leq t \leq b$*, level* x *beats* t*; and for all* $t \geq b$*, level* y *beats* t*; thus there is no Condorcet winner.*

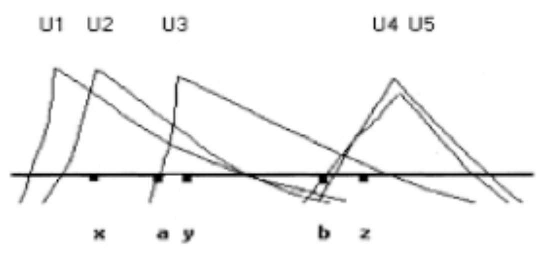

Figure 1: Absence of Condorcet Winner

The above examples are in sharp contrast with Fishburn's Theorem on single plateaud preferences. Introducing indifferences in the "central" part of a single peaked preferences complicates life, but allows to basically maintain the spirit of Black's median voter result. Now we see that introducing indifferences in the "tails" of single peaked preferences causes much deeper changes in the workings of the majority rule.

The fact that additional indifferences could cause trouble has always been well known. Even the excellent textbook on microeconomics by Mas-Colell, Winston and Green contains an exercise to this effect (exercise 21.D.14). What I find quite remarkable is that one does not need any complicated distributions of indifferences to create anomalies: it is enough with the simple changes induced by an outside option.

3.2 Positive Results on Strategy Proofness

Another striking contrast arises. There has been a long tradition in social choice emphasizing the strong connections between those domains which admit a non-dictatorial strategy proof social choice rule, and those for which it is possible to define an Arrowian social welfare function (see Satterthwaite 1975 for the case of unrestricted domains, Kalai and Muller 1977, and Muller and Satterthwaite 1985 for general domains, Barberà, Gul and Stacchetti 1993 for generalized single peaked domains).

Yet, in the case of single peaked preferences with outside options, a great gap arises. The preceding examples show that majority fails to define a social welfare function on this domain, and suggest that it will be hard to define any Arrowian one. Yet, the following result, also due to Cantala, proves that there do exist quite a few non-dictatorial strategy proof social choice function in the domain, and it is possible to characterize them.

Theorem 6 *Let $f : O \to A$ be a social choice function on the set of single peaked preferences with outside options. Then, f is strategy proof and efficient if and only if there exists a set of parameters $\{a_s\}_{S \subseteq N}$ on $\{a, b\}$, with $a_\phi = b$ and $a_N = a$ such that, for all $\succcurlyeq \in O(>)$, $f(\succcurlyeq) = \min_{S \subseteq N} \{\max_{i \in S} \{p(u_i), a_s\}\}$.*

Notice that this is a subclass of all the efficient rules characterized by Moulin, but still a rich one.

Notice also that, if we dropped efficiency, the class would grow much larger, beyond that of non-efficient rules characterized by Moulin, because in this case the additional information about preferences which is contained in the upper and lower limits of the consumption sets of individuals can be used to create new social choice functions.

4. From Troublesome Indifferences to Helpful Ones

We now turn to a third type of models where indifferences arise. However, unlike what happened in the preceding section, now it will be the presence of such indifferences that will suggest the possibility of defining families of preference profiles under which the use of the simple majority rule always produces a Condorcet winner. Rather than going through a formal discussion, I will present a number of examples which are used in Barberà and Ehlers (2007).

In that paper, we remark that there exist several families of domains where the existence of Condorcet winners can be guaranteed, provided that the distribution of indifferences over alternatives across individuals satisfies rather specific conditions. An important feature of these domains is that they are personalized. That is, the preferences which are admissible for each voter are not necessarily the same than those admissible for others. For details and formal statements, I refer to the above mentioned paper.

For a first example, consider a faculty board having to elect a department chair. Any faculty member is eligible, and is entitled to decline the position if eventually elected. It is therefore interesting to establish not only a winner, but also a list of alternates in case of resignation. Suppose the faculty is objectively divided into two groups, say theorists and applied, that each voter considers his or her own case as separate from the rest and that each voter treats all other candidates as members of one of the two groups, being indifferent between any two theorists or between any two applied people (other than his or her own). All possible orderings of the three personalized indifference classes (oneself, theorists and applied, minus eventually oneself) are admissible. For an even number of faculty members our conditions hold, and the use of the majority rule guarantees that the derived pairwise comparisons between candidates satisfy quasi-transitivity. If the number of faculty members is odd, then majority rule violates quasi-transitivity.

A similar example arises if, instead of a partition, we start from a linear order of the candidates. Each agent has then one neighbor to his left, and one to his right (except for those at the extremes, who only have one neighbor). We allow each agent to freely order himself and its immediate neighbors. Some agents may prefer to be elected rather than seeing his neighbors elected. Others may prefer their neighbor to the right (or left) to be the winner. Likewise, second and third positions in the ranking are free. Hence, our preferences allow for free triples of alternatives. Yet, we also assume that agents cannot clearly distinguish between a victory by their neighbor in the right and the victory of any other candidate in the same direction (this maybe due to myopia, or else result from rational calculations). He is indifferent among all candidates to his right, and also among all candidates to his left.

In this example, then, each agent has (at most) three indifference classes: all people to the left of the voter form a class, all people to the right of the voter form a second class, and the voter himself is a third class. These classes can be ordered in any possible way. One can prove that if all agents have preferences of this type, the majority relation associated to any profile of opinions by any number of voters is quasi-transitive (i.e., the strict majorities among alternatives respect transitivity). Again this is sufficient for the existence of Condorcet winners.

The example can be extended. The linear structure is unnecessarily narrow. Agents may be at the nodes of any tree. Therefore, each agent may have a different number of neighbors, as many as the number of branches starting from or arriving to that node. An agent with k neighbors may now freely rank $k + 1$

classes of alternatives, including oneself as a singleton class. Each of the remaining classes includes one neighbor and all other candidates which are further out than this neighbor within the tree structure. This is a very substantial extension of the domain: very perceptive people can have many neighbors and thus freely rank many classes of candidates. Other agents may be restricted by their positions to only rank a few groups. Many structures are allowed, provided they can be represented by a tree, of any form.

A context where such a tree structure gives rise to preferences of this sort is related to a problem discussed by Demange (2004). This author studies the distribution of profits from cooperation in games with hierarchies. Hierarchies are described by a tree, which describes connections between agents, and by a specific individual, among all the agents, who plays the role of the principal. She shows that if only coalitions which are properly connected can form (she calls them teams), then the (restricted) core of the cooperative game among these agents is nonempty and easy to describe. The admissible teams (and thus, the resulting core distribution) depend on the tree and also on the specific agent who plays the role of principal. One may extend the analysis of Demange by separating these two ingredients, and by allowing all possible hierarchies which arise from the same tree, as the principal changes. In Demange's interpretation, the tree expresses possible channels of communication among agents. Suppose that these channels are technologically determined, but that the directions of hierarchical communication may be chosen. For example, all agents may vote on who is going to play the role of the principal. Their preferences will depend on the payoffs that they will get in the core, depending on who is the principal (for a given tree). It turns out that agents will get the same payoff for all principals who are on the same branch away from some of their immediate neighbors. One can prove that, in that case, the majority rule would always determine (at least) one winner if agents in that context would vote for a principal. That is, preferences induced by the proposed extension of Demange's model satisfy the following condition: given a triple $fa; b; cg$, there is exactly one agent for whom $fa; b; cg$ is free, namely the agent who is located at the median of a, b and c.

These examples are rather special, and leave room for many other applications. Yet, the examples already suggest that the reasons for agents to be indifferent among sets of alternatives may come from different sources[7], ranging from technological to purely subjective, and may be associated to asymmetries of different sorts, which can be formalized through networks, rankings or partitions.

5. Some Final Remarks

Most social choice theorists have always been aware of the points I have stressed in this essay. For example, the classical work of Sen and Pattanaik (1969) on

[7] In certain contexts it may be attractive to distinguish between indifference and incompleteness when modelling the inability or the lack of willingness of some agents to express a strict preference between some pairs of alternatives. But this distinction would not lead to any substantial or operational gain in our context, since expressing indifference or not expressing any preference are equivalent for the workings of majority rule.

extremal restrictions is careful to consider indifferences, Inada's (1964; 1969) notion of dichotomic preferences builds on the notion that voters may have large indifference classes, and the very idea of single crossing preferences (Mirrlees 1971) is best understood when we think of indifference curves.

I also want to acknowledge that my choice of examples is biased, and one could have used many others. for example, there are interesting extension of results based on single peakedness to the case where alternatives are not linearly ordered, but are in the nodes of a tree. Demange (1982) and Schummer and Vohra (2002) provide results on Condorcet winners and strategy proof rules that parallel those we have described, and the introduction of indifferences in those cases would have a similar impact. Indifferences also create additional difficulties in related contexts which I have not described. For example, one must be careful with Maskin monotonicity, an important condition in implementation theory, whose implications for some types of preferences allowing for indifferences are stronger than it is usually thought. For a discussion, I refer the reader to Jackson's (2001) excellent primer on implementation (Section 2.3).

Yet, I hope these warnings and remarks can be useful for those practitioners who use these notions without so much attention for issues that are often thought of as details. They can be helpful in two ways. One, as a warning to the reader that some technical problems may lie ahead, when translating results from one application to others that look similar but are not identical. The other aspiration of usefulness is more ambitious. The discovery of new and relevant domain restrictions must come from those who use the general results with specific purposes, and find innovative methods to avoid old problems. When in need to escape from problems, people find new ways out. I wanted to call attention to the problems caused by indifferences in order to stimulate new solutions[8].

After Black, other important authors, like Grandmont (1978) and Roberts (1977) have proposed alternative restrictions which are useful in many contexts. In a recent paper (Barberà and Moreno 2007), it is shown that (in a world with indifferences), single peakedness, intermediateness, single crossing and adherence have a common root. I have found it quite interesting to explore these connections, but this was only possible because others had developed these different concepts. I hope to see others emerging in the future.

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[8] As a matter of fact, new discoveries about domain restrictions are still being made, even in the most classical context of linear preferences. Ballester and Haeringer 2007 have recently provided necessary and sufficient local conditions on preference profiles for them to be globally single peaked.

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